LEPTON NUMBER CONSERVATION

AND

DOUBLE BETA DECAY

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DEDICATION

To Josef Kvasnica
The Alpha
If not the Omega

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TABLE OF CONTENTS

APPENDIX

LIST OF TABLES

LIST OF ILLUSTRATIONS

SUMMARY

The discovery of the neutrino and the observation that parity is not conserved in weak interactions have played an important role in the development of the theory of beta decay. Within experimental error ordinary (single) beta decay has been shown to proceed via a local four-fermion interaction which can be expressed as the product of a hadronic weak current and a leptonic weak current. Nuclear double beta decay is considered as a second order effect of the same interaction which gives rise, in first order, to single beta decay. It was originally anticipated as a two-neutrino process, i.e. one in which two neutrinos as well as two electrons are emitted. Investigations into the properties of the neutrino have led, however, to speculation about a possible neutrinoless mode of decay. Considerable theoretical interest attaches to this possibility which, if realized, would provide irrefutable evidence against the principle of lepton number conservation. Several attempts to observe double beta decay experimentally have been made and it is now established that this phenomenon does occur. Unfortunately, most of the experiments only detect the daughter nucleus from double beta decay, and thus furnish no information on whether neutrinos are emitted or not. Only very recently has some attempt been made to remedy this deficiency.

An examination of the processes of double beta decay (neutrinoless and two-neutrino decays) plays an important part in the clarification of the mechanism of weak interactions, and this provides the moti-

vation for this investigation. In order to accomodate double beta decay without emission of neutrinos, the leptonic weak current must be modified so as to include terms with both helicity projection operators ($1 + \gamma_5$ and $1 - \gamma_5$) whose relative weight is a factor η known as the lepton non-conservation parameter. It is our purpose to determine the magnitude of this parameter and to see whether we can furnish any additional information which bears on the question of nuclear double beta decay.

There are a number of uncertainties which mitigate against a precise evaluation of these quantities. In the first place, the energy gaps between the initial nuclear state and those intermediate nuclear states which contribute (in the allowed approximation) to the decay rate are, in general, not known. Secondly, it is very difficult to calculate the magnitudes of the nuclear matrix elements involved, and in most cases it has been necessary to fall back on the traditional method of approximating them. Thirdly, the experimental life-time of the double beta decay is needed for an estimation of the parameter η, and this is not always known precisely.

We have concentrated on the three decays

$$^{130}Te_{52} \rightarrow {}^{130}Xe_{54}$$

$$^{48}Ca_{20} \rightarrow {}^{48}Ti_{22}$$

$$^{128}Te_{52} \rightarrow {}^{128}Xe_{54}$$

for which the most reliable data is available. Using the known experimental lifetime of the ^{130}Te decay and the lower limits on the half-lives of the no-neutrino and two-neutrino modes of ^{48}Ca decay, we have obtained the following results :

$$0.23 \times 10^{-4} < \eta < 0.53 \times 10^{-3}$$

$$2.1 < \delta\left({}^{130}\mathrm{Te}\right) < 2.68$$

$$24.24 < T\left({}^{128}\mathrm{Te}\right) < 24.95$$

where δ is the average energy difference between the initial nuclear state and those intermediate nuclear states with spin 0 or 1, and the half-life of the double beta decay is written as 10^{T} years.

NOTATION

Throughout this dissertation we have used the Pauli-Dirac metric

$$A = (\vec{A}, iA_o)$$

three-vectors being denoted by an arrow. The scalar product of two four-vectors A and B is

$$A \cdot B = A_\mu B_\mu = \vec{A} \cdot \vec{B} + A_4 B_4$$

where $$A_4 = iA_o$$

Unless otherwise stated, Latin indices i, j ; a, b etc run from 1 to 3. Greek indices λ, μ ; α, β etc run from 1 to 4.

With this metric the gamma matrices are all Hermitian :

$$\gamma_\mu = \gamma_\mu^\dagger$$

The Einstein convention for repeated suffixes in summations has been used throughout, e.g.

$$A_{\lambda\alpha} B_{\mu\alpha} \equiv \sum_\alpha A_{\lambda\alpha} B_{\mu\alpha}$$

The symbols * and † denote complex conjugation and Hermitian conjugation, respectively, while $\sim$ denotes transposition of a matrix.

CHAPTER I

INTRODUCTION

The development of our knowledge of the beta decay of radioactive nuclei can be divided into two distinct historical stages. The start of each of them was heralded by important discoveries which necessitated a fundamental review of earlier concepts concerning this phenomenon.

In 1931 Pauli[1] advanced the hypothesis of the existence of a neutral, massless particle, called the neutrino. This hypothesis afforded a way out of the apparent contradiction between the conservation laws and the beta decay phenomena. Using this hypothesis, Fermi[2] in 1934 developed a beta decay theory which explained a number of the observed features of beta decay.

The second important discovery was the observation that parity is not conserved in weak interactions. The detailed investigation of this question, culminating in the famous experiments of Wu et al[3] was instigated by the problem of K^+ meson decay (the θ - τ puzzle) : although all the physical properties of the primary particle in θ decay (two-pion final state) and in τ decay (three-pion final state) were identical, analysis indicated opposite parities for the final states. This dilemma prompted Lee and Yang[4] to undertake a systematic examination of all the experimental knowledge concerning parity violation. They found that no experiments had ever been designed specifically to test whether parity is conserved in weak interactions, and that there was no experimental data containing information of relevance to this question. The experiments of Wu et al. proved conclusively that parity is not conserved in the beta decay of radioactive nuclei.

Soon after the announcement of the Fermi theory, Geoppert-Mayer[5] published a paper devoted to a theoretical investigation of the possible properties of the neutrino. This paper contained the first statement of the hypothetical feasibility of double beta decay, envisaged as a transition involving the emission of two electrons and two neutrinos. As is well known, there are two types of (single) beta decay : electronic decay

$$n \rightarrow p + e^- + \bar{\nu} \qquad (1.1a)$$

and positronic decay

$$p \rightarrow n + e^+ + \nu \qquad (1.1b)$$

This immediately raises the question whether the neutral leptons emitted in these two processes are identical. In 1937 Majorana[6] showed that if the neutrino is its own antiparticle ($\nu \equiv \bar{\nu}$), then the deductions of the beta decay theory remain unchanged, and Racah[7] noted that in this case a neutrinoless double beta decay becomes possible. Two years later Furry[8] investigated the neutrinoless mode of double beta decay which he envisaged as proceeding in the following manner : the initial nucleus emits an electron and goes over into a virtual intermediate state plus a virtual neutrino which, interacting with the intermediate nucleus, "induces" its decay, with emission of a second electron, into the final nucleus, and is itself absorbed. Although he made an incorrect estimate of the probability of this double beta transition, the mechanism outlined above is still used today.

Lepton Number Conservation And Double Beta Decay

The law of lepton number conservation was proposed by Konopinski and Mahmoud[9] in order to explain the non-occurrence of certain decay processes, e.g.

$$\bar{\nu} + n \not\rightarrow p + e^- \qquad (1.2a)$$

This is confirmed by the observation, using $\bar{\nu}$'s from a reactor[10], that

$$\bar{\nu} + {}^{37}Cl_{17} \not\rightarrow {}^{37}Ar_{18} + e^- \qquad (1.2b)$$

The leptons ($e^{\pm}$, ν, $\bar{\nu}$) are assigned a lepton number L according to the following scheme*:

$$L(e^-) = +1 \quad \text{(by definition)} \qquad (1.3)$$

$$L(\text{particle}) = -L(\text{antiparticle}) \qquad (1.4)$$

$$L(\text{non-lepton}) = 0 \qquad (1.5)$$

and the law states that the algebraic sum of the lepton numbers is conserved in all reactions. If lepton number is a good quantum number, then two-neutrino double beta processes, e.g.

$$^{130}Te_{52} \rightarrow {}^{130}Xe_{54} + e^- + e^- + \bar{\nu} + \bar{\nu} \qquad (1.6)$$

* Since we are not concerned here with muonic processes, the muon lepton numbers have been omitted from the discussion.

would be permitted. On the other hand, no-neutrino processes such as

$$^{130}Te_{52} \rightarrow {}^{130}Xe_{54} + e^- + e^- \tag{1.7}$$

would be forbidden. The existence of a neutrinoless mode of decay would be a clear instance of the violation of lepton number conservation, and it is for this reason that such interest has been generated in the search for double beta decays without emission of neutrinos.

The double beta ($\beta\beta$) decay process consists of the spontaneous disintegration of a nucleus (A, Z − 2) into the nucleus (A, Z), the transition being accompanied by the emission of two electrons*, with or without neutrinos. Such transitions are made by nuclei which are less stable when one of their neutrons is replaced by a proton, but which gain in stability when two of their neutrons are replaced by protons. This is understoood to be a consequence of strong, attractive "pairing forces" between like nucleons, and it is to be expected that both the parent and the daughter nuclei are of the even-even type. The initial state in any $\beta\beta$ process, being the ground state of an even-even nucleus, is characterized by zero spin and even parity, 0^+. The final state is also expected to be the ground state of the daughter nucleus, which is again 0^+.

The intermediate nucleus (A, Z − 1) is an odd-odd nuclear isobar in which a neutron from the initial nucleus has been changed into a proton. The nuclear energy levels of (A, Z − 1) are generally higher than $E_i + mc^2$, so that the initial nucleus is stable against single β decay. It may happen,

* We shall restrict ourselves to the discussion of $\beta^-\beta^-$ decays.

however, that this single β decay $(A, Z-2) \rightarrow (A, Z-1)$ is possible energetically, perhaps to the ground state of $(A, Z-1)$, but is so strongly inhibited by the associated large spin change that the $\beta\beta$ decay $(A, Z-2) \rightarrow (A, Z)$ is actually more probable. For example, the $\beta\beta$ decay

$$0^+ \text{ ground state of } {}^{48}Ca_{20} \rightarrow 0^+ \text{ ground state of } {}^{48}Ti_{22}$$

is expected to proceed at a much faster rate than the energetically possible, but sixth forbidden single β decay

$$0^+ \text{ ground state of } {}^{48}Ca_{20} \rightarrow 6^+ \text{ ground state of } {}^{48}Sc_{21}$$

the half-life of which is, according to Feenberg[11] $\sim 10^{25}$ years.

The "energy release" in a $\beta\beta$ decay process is equal to the maximum kinetic energy available to either of the emitted electrons, assuming that the recoil energy of the daughter nucleus is negligible.

Experimental Evidence

Several experimental investigations of the $\beta\beta$ radioactivity of suitable isotopes have been carried out within the last two decades, using a variety of techniques, which include :

(i) chemical extraction of the daughter element (A, Z) and a study of its abundance within sources of the parent element.

(ii) detection in a cloud chamber of nuclear emulsion of e_1, e_2 tracks emanating from a common point.

(iii) counter detection of e_1, e_2 time-coincidences from a $\beta\beta$ decaying element.

These methods have been described in detail by a number of authors[12,13] where references to the individual experiments may be found.

Experiments of type (i) above detect only the presence of the daughter nucleus from $\beta\beta$ decay and yield only the decay rate or half-life of the decay. They give no indication whether neutrinos are emitted or not, and hence whether lepton conservation is violated. The identification of the no-neutrino and two-neutrino modes of decay can be made by observing the sum spectrum of the two-electron energy. In neutrinoless decay the total energy of the two emitted electrons must be equal to the available decay energy ; and the spectrum of the total electron energy should have the form of a narrow peak, whose position corresponds to the decay energy. In the two-neutrino case, however, the decay energy will be statistically distributed among all four leptons and the electron sum energy spectrum will have a broad distribution, peaking at about half the available energy.

Convincing evidence of $\beta\beta$ decay has been obtained by mass-spectrometric analyses[14-17] of Tellurium and Selenium ores of known ages. Kirsten et al[15] have analyzed the isotopic composition of Xenon extracted from a native Tellurium ore from the Good Hope mine in Colorado and have reported a ratio ^{130}Xe(excess)/ ^{130}Xe(atmospheric) > 50. This large excess of ^{130}Xe, unaccompanied by any anomalies, can be explained only by the $\beta\beta$ decay of ^{130}Te. On the basis of the age of the ore and the concentrations of ^{130}Te and ^{130}Xe in it, they have estimated the half-life of ^{130}Te to be $10^{21.34 \pm 0.12}$ years. In an earlier experiment using different Tellurium ores Takaoka and Ogata[16] found an excess of ^{128}Xe, but these authors have cautioned that it is difficult to attribute the excess ^{128}Xe entirely to the $\beta\beta$ decay of ^{128}Te because of a small, persistent background in the mass spectrometer, which disturbed exact measurements at ^{128}Xe. A ^{128}Te half-life of $10^{22.5 \pm 0.5}$ years was obtained on the assumption that all the excess ^{128}Xe was due to ^{128}Te $\beta\beta$ decay. Unfortunately, as explained above, these experiments do not give any indication whether neutrinos are emitted, and hence whether lepton number is conserved. Only an examination of the electron sum-energy spectrum can do this.

An attempt was made recently by Bardin et al.[(18)] to measure the continuous electron sum-energy spectrum of the two-neutrino decay of ^{48}Ca and the monoenergetic sum-spectrum of the neutrinoless decay. The experiment was performed in a deep salt mine, using a streamer chamber in a magnetic field. Only one event was seen which might be construed as neutrinoless $\beta\beta$ decay, implying a half-life of $> 2.0 \times 10^{21}$ years. A half-life limit of $> 3.6 \times 10^{19}$ years for the two-neutrino mode of ^{48}Ca $\beta\beta$ decay was also obtained.

Single Beta Decay

Most of our knowledge concerning the weak interactions has been obtained from studies of the decay processes of the elementary particles. We give here a brief discussion of single β decay with a view to establishing the V – A (vector minus axial-vector) character of the couplings involved in the four-fermion interaction, and ultimately to writing this interaction in current-current form.

The prototype of the β decay process is the reaction

$$n \rightarrow p + e^- + \bar{\nu}$$

although, of course, most of the experimental information on this process is gathered from observations of the β decay of complex nuclei.

Allowed Selection Rules

The spatial part of the wave function of the outgoing particles is of the form

$$e^{-i(\vec{k}_e + \vec{k}_\nu)\cdot\vec{r}} = e^{-i\vec{k}\cdot\vec{r}}.$$

The momentum of the leptons typically corresponds to a de Broglie wavelength of about 10^{-11} cm and a useful size for r is the nuclear radius, which is of the order of 10^{-13} cm. Thus $\vec{k}\cdot\vec{r} \sim 10^{-2}$ and we make the approximation

$$e^{-i\vec{k}\cdot\vec{r}} \sim 1.$$

This result implies that the outgoing leptons carry off zero orbital angular momentum ℓ, since $e^{-i\vec{k}\cdot\vec{r}}$ can be expanded in a series of spherical harmonics containing Bessel functions which behave like $(\vec{k}\cdot\vec{r})^{\ell}$ for small values of $\vec{k}\cdot\vec{r}$. Transitions in which the leptons carry off zero orbital angular momentum ($\ell = 0$) are known as allowed transitions ; non-zero values of ℓ correspond to forbidden transitions, the degree of forbiddenness being equal to the number of units of orbital angular momentum carried away by the electron-neutrino pair. As may be expected from the foregoing, the rates for forbidden transitions are considerably slower than for allowed transitions.

The allowed transitions may be classified according to the resultant intrinsic spin $\vec{S}$ of the electron-neutrino pair. Transitions in which the leptons are emitted in the singlet state ($\vec{S} = \vec{0}$) are known as Fermi transitions ; those in which the leptons are emitted in the triplet state

($\vec{S} = \vec{1}$) are known as Gamow-Teller transitions. The allowed selection rules follow from a consideration of the overall angular momentum balance

$$\vec{I}_i - \vec{I}_f = \vec{s}_e + \vec{s}_\nu = \vec{S}$$

where $\vec{I}$ is the nuclear spin. For Fermi transitions we must clearly have $\vec{I}_i = \vec{I}_f$. This selection rule,

$$\Delta I \equiv |I_i - I_f| = 0$$

(together with the requirement of no change in the nuclear parity, as for all allowed transitions), is known as the Fermi selection rule. For triplet emissions the angular momentum balance is

$$\vec{I}_i = \vec{I}_f + \vec{1} .$$

The vector sum on the right may have any of the magnitudes $I_f + 1$, I_f or $I_f - 1$, so we obtain the Gamow-Teller selection rule

$$\Delta I = 0, 1 \ (\text{no } \vec{0} \rightarrow \vec{0})$$

The additional proviso in parentheses arises because such transitions are clearly incompatible with angular momentum conservation.

The Beta Decay Interaction

The beta decay Hamiltonian as originally conceived by Fermi may be written in the form

$$\mathcal{H} = C\left(\overline{\Psi}_p \gamma_\lambda \Psi_n\right)\left(\overline{\Psi}_e \gamma_\lambda \Psi_\nu\right) \tag{1.8}$$

where C denotes the coupling strength. This expression does not, however, represent the most general four-fermion interaction one can construct.

There are five Dirac bilinear covariants, of the form $\bar{\Psi}\,\Gamma_i\,\Psi$, where Γ_i ($i = S, V, T, A, P$) are the scalar, vector, tensor, axial-vector and pseudoscalar Dirac operators. We will generalize (1.8) to include all possible types of Dirac bilinears :

$$\mathcal{H} = \sum_i C_i\left(\bar{\Psi}_p \Gamma_i \Psi_n\right)\left(\bar{\Psi}_e \Gamma_i \Psi_\nu\right) \tag{1.9}$$

It will be noted that the interaction (1.9) contains only parity conserving terms. The question of parity violation will be taken up at a later stage. In the meantime, a great deal of information can be obtained about the various types of coupling without recourse to a parity-violating formulation. The classical picture embodied in (1.9) can still be used to interpret the results of many important beta decay experiments, such as beta decay lifetimes, energy spectra and electron-neutrino angular correlations.

As far as nuclear beta decay is concerned, we may neglect the recoil of the nucleus (the energy release is small compared with the nucleon mass) and the nucleon spinors may be treated non-relativistically. It will be recalled that in this approximation a Dirac four-component spinor may be written in terms of its large and small components

$$\Psi = \begin{pmatrix} \Psi_L \\ \Psi_S \end{pmatrix}$$

where Ψ_L and Ψ_S are two-component spinors and Ψ_S vanishes in the limit of small velocities $\frac{v}{c} \to 0$. We now examine the different types of coupling in the light of the non-relativistic approximation. For scalar coupling the nucleon bilinear is of the form

$$\overline{\Psi}\,\Psi = \left(\Psi_L^\dagger \quad -\Psi_S\right)\begin{pmatrix}\Psi_L\\ \Psi_S\end{pmatrix} \sim \Psi_L^\dagger\,\Psi_L\ .$$

For vector coupling, we have

$$\overline{\Psi}\,\gamma_\lambda\,\Psi \equiv \left(\overline{\Psi}\,\vec{\gamma}\,\Psi,\ \overline{\Psi}\,\gamma_4\,\Psi\right) \sim \left(\vec{0},\ \Psi_L^\dagger\,\Psi_L\right)\ .$$

It is evident that the scalar and vector terms are incapable of inducing any nuclear spin change (the spin operator $\vec{\sigma}$ does not appear). These terms are associated with Fermi transitions. For the remaining types of coupling we obtain

$$A \rightarrow \left(\Psi_L^\dagger\,\vec{\sigma}\,\Psi_L,\ \ 0\right)$$

$$T \rightarrow \left(\Psi_L^\dagger\,\vec{\sigma}\,\Psi_L,\ \ 0\right)$$

$$P \rightarrow 0.$$

The axial-vector and tensor terms can induce a nuclear spin change, and are associated with Gamow-Teller transitions. The pseudoscalar term is neglected. Considering that beta decays can be a mixture of Fermi and Gamow-Teller transitions, we see that there are both

S and/or V couplings (Fermi transitions)

A and/or T couplings (Gamow-Teller transitions)

Further information on the nature of the couplings may be obtained by considering the most general (Fermi plus Gamow-Teller transitions) type of β decay of unpolarized nuclei. Detailed calculations[19] yield the result that the energy spectrum of the emitted electrons is of the form

$$N(E)\ dE \sim A + \frac{B}{E} \tag{1.10}$$

where

$$A = (C_S^2 + C_V^2)\,|M_F|^2 + (C_A^2 + C_T^2)\,|M_{GT}|^2 \tag{1.11}$$

$$B = C_S C_V |M_F|^2 + C_T C_A |M_{GT}|^2 \tag{1.12}$$

and M_F and M_{GT} are the Fermi and Gamow-Teller matrix elements, respectively. A comparison of this result with experiment shows that $B = 0$ for all β decay spectra. From the existence of pure Fermi transitions, e.g.

$$^{14}O \rightarrow {}^{14}N^* + e^+ + \nu$$

and pure Gamow-Teller transitions, e.g.

$$^{12}B \rightarrow {}^{12}C + e^- + \bar{\nu}$$

we may infer that C_S and C_V cannot both be zero ; and that C_A and C_T cannot both be zero. These conclusions amount to a selection rule which may be stated as follows :

Fermi transitions : either S or V coupling

G. T. transitions : either A or T coupling

The exact nature of the coupling is obtained from the electron-neutrino correlation. This can be shown[20] to be

$$1 + a\,\frac{v_e}{c}\cos\theta_{e,\nu}$$

where $$a = \frac{(C_V^2 - C_S^2)|M_F|^2 + \frac{1}{3}(C_T^2 - C_A^2)|M_{GT}|^2}{(C_V^2 + C_S^2)|M_F|^2 + (C_T^2 + C_A^2)|M_{GT}|^2} . \qquad (1.13)$$

A study of the directional correlation between electron and neutrino (really the recoil of the nucleus) for various β emitters yields the following results : for pure Fermi transitions[21]

$$a = \frac{C_V^2 - C_S^2}{C_V^2 + C_S^2} = 1 \text{ (expt)} \rightarrow \text{V coupling,}$$

while for pure Gamow-Teller transitions[21,22]

$$a = \frac{1}{3}\frac{C_T^2 - C_A^2}{C_T^2 + C_A^2} = -\frac{1}{3} \text{ (expt)} \rightarrow \text{A coupling.}$$

We see, therefore, that nuclear beta decay proceeds via the VA interaction.

We now return to the question of parity violation. In β decay there are several independent vectors or axial-vectors, e.g. $\vec{p}_e$, $\vec{p}_\nu$, $\vec{I}$ (where $\vec{I}$ is the spin of the parent nucleus). These vectors may be observed in various combinations in different experiments, some combinations being true scalars and some pseudoscalars. In a parity-conserving process all measured quantities must depend on true scalars. On the other hand, if a measured quantity could be shown to depend on a pseudoscalar entity, such as $\vec{p}_e \cdot \vec{I}$, this would demonstrate the violation of parity. In the old beta decay experiments, in which unpolarized nuclei were used (no $\vec{p} \cdot \vec{I}$), the only observable quantities were essentially $\vec{p}_e$ and $\vec{p}_\nu$, and clearly no pseudoscalar can be formed from these. It is for this reason that, despite the voluminous experimental data on β decay, no light was shed on the question of parity violation. Indeed, this possibility was not even discussed, nor was any attempt made to observe the electron polarization $(\vec{\sigma}\cdot\vec{p})$. At the suggestion of Lee and Yang[4] the famous ^{60}Co experiment was performed by Wu et al.[3]. This was done by lining up the ^{60}Co nucleus and measuring the asymmetry of the electron momentum with respect to the direction of the nuclear spin $\vec{I}$. The angular distribution was found to be of the form

$$W(\theta) \sim A\left(1 + \frac{B}{A}\,\vec{p}_e\cdot\vec{I}\right) \; ; \; \frac{B}{A} \neq 0, \tag{1.14}$$

which indicates that parity is not conserved in beta decay.

The above result necessitates a modification of (1.9) to include parity violating terms :

$$\mathcal{H} = \sum_i C_i\left(\overline{\Psi}_p\Gamma_i\Psi_n\right)\left(\overline{\Psi}_e\Gamma_i\Psi_\nu\right) + C_i'\left(\overline{\Psi}_p\Gamma_i\Psi_n\right)\left(\overline{\Psi}_e\Gamma_i\gamma_5\Psi_\nu\right)$$

$$= \sum_i \left(\overline{\Psi}_p\Gamma_i\Psi_n\right)\left[C_i\overline{\Psi}_e\Gamma_i\Psi_\nu + C_i'\overline{\Psi}_e\Gamma_i\gamma_5\Psi_\nu\right] \tag{1.15}$$

or

$$\mathcal{H} = \sum_i \left(\overline{\Psi}_p\Gamma_i\Psi_n\right)\left[\frac{C_i + C_i'}{2}\,\overline{\Psi}_e\Gamma_i(1+\gamma_5)\Psi_\nu + \frac{C_i - C_i'}{2}\,\overline{\Psi}_e\Gamma_i(1-\gamma_5)\Psi_\nu\right] \tag{1.16}$$

The Two-component Neutrino Theory

The Dirac equation for a massless spin $\frac{1}{2}$ particle may be written in the form

$$(\vec{\alpha}\cdot\vec{p})\Psi = E\,\Psi \tag{1.17}$$

where Ψ is a four-component spinor

$$\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

and φ and χ are two-component spinors. Thus we have

$$E \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

using the Dirac representation of the gamma matrices. This gives

$$E\,\varphi = (\vec{\sigma}\cdot\vec{p})\,\chi$$

$$E\,\chi = (\vec{\sigma}\cdot\vec{p})\,\varphi$$

Adding and subtracting these equations, we get

$$(\vec{\sigma}\cdot\vec{p})\,\xi = -E\,\xi \tag{1.18}$$

$$(\vec{\sigma}\cdot\vec{p})\,\eta = +E\,\eta \tag{1.19}$$

where
$$\xi \equiv \varphi - \chi$$

$$\eta \equiv \varphi + \chi$$

Equations (1.18) and (1.19) are known as the Weyl equations for massless, spin $\frac{1}{2}$ particles. Considering only positive energy solutions ($E = +|\vec{p}|$), the Weyl equations yield

$$(\vec{\sigma}\cdot\vec{n})\,\xi = -\,\xi \tag{1.20}$$

$$(\vec{\sigma}\cdot\vec{n})\,\eta = +\,\eta \tag{1.21}$$

where $\vec{\sigma}\cdot\vec{n} \equiv \dfrac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}$ is the helicity operator. Thus a massless, spin $\frac{1}{2}$ particle is completely polarized, either left or right. We see that ξ describes a particle whose spin is always antiparallel to its momentum (negative helicity), while η describes a particle whose spin is always parallel to its momentum (positive helicity).

We now introduce the positive and negative "chiral projections"

$$\Psi_{\pm} = \frac{1}{2}(1 + \gamma_5)\Psi$$

which are both eigenstates of γ_5 :

$$\gamma_5\Psi_{\pm} = \pm\Psi_{\pm}$$

Now in the Dirac representation of the gamma matrices we may write

$$\Psi_{+} = \frac{1}{2}(1 + \gamma_5)\Psi = \frac{1}{2}\begin{pmatrix} \varphi - \chi \\ -(\varphi - \chi) \end{pmatrix}$$
$$\Psi_{-} = \frac{1}{2}(1 - \gamma_5)\Psi = \frac{1}{2}\begin{pmatrix} \varphi + \chi \\ \varphi + \chi \end{pmatrix} \tag{1.22}$$

so that if we project with the positive (negative) chirality operator, we deal essentially with the two-component spinor $\xi = \varphi - \chi$ ($\eta = \varphi + \chi$). We have just seen that ξ (η) describes a negative (positive) helicity particle, i.e. the term $\frac{1}{2}(1 + \gamma_5)\Psi$ describes a particle with spin pointing opposite to the direction of motion ; and the term $\frac{1}{2}(1 - \gamma_5)\Psi$ describes a particle whose spin is parallel to its direction of motion.

Helicity of the Neutrino

The neutrino is defined as the neutral lepton emitted in β^+ decay

$$p \rightarrow n + e^+ + \nu.$$

Now since the absorption of a particle (e^-) and the emission of the anti-particle (e^+) are equivalent processes, the lepton emitted in electron capture must also be a neutrino,

$$p + e^- \rightarrow n + \nu.$$

The helicity of the neutrino was determined[23] from a study of the electron capture process

$$^{152}\mathrm{Eu} + e^- \rightarrow {}^{152}\mathrm{Sm}^* + \nu \rightarrow {}^{152}\mathrm{Sm} + \gamma + \nu.$$

The chief point of this experiment is that one can select those γ rays from the decay of the excited state which go opposite to the ν direction (i.e. in the direction of the recoil nucleus) by having them resonance scatter from a target of ^{152}Sm. It can be shown from considerations of overall angular momentum conservation that the helicity of the downward γ ray will be the same as that of the upward neutrino, irrespective of whether this (the neutrino helicity) is positive or negative. The experiment established the helicity of the γ ray as negative, indicating that the neutrino is left-handed.

Going back to equation (1.16), we see that since only the negative helicity neutrino states are realized in nature, the $(1 - \gamma_5)\Psi_\nu$ terms must be dropped. This implies that

$$C_i = C_i' \equiv \frac{G_i}{\sqrt{2}} \tag{1.23}$$

and we obtain

$$\mathcal{H} = \sum_i \frac{G_i}{\sqrt{2}} \left(\overline{\Psi}_p \Gamma_i \Psi_n\right)\left[\overline{\Psi}_e \Gamma_i (1 + \gamma_5)\Psi_\nu\right] \tag{1.24}$$

where the summation runs over V and A couplings. The usual vector and axial-vector operators are γ_λ and $i\gamma_\lambda\gamma_5$ respectively (the factor i is inserted so as to make Γ_i Hermitian), so we obtain

$$\mathcal{H} = \frac{G_V}{\sqrt{2}} \left(\overline{\Psi}_p \gamma_\lambda \Psi_n\right)\left[\overline{\Psi}_e \gamma_\lambda (1 + \gamma_5)\Psi_\nu\right] +$$

$$+ \frac{G_A}{\sqrt{2}} \left(\overline{\Psi}_p i\gamma_\lambda\gamma_5 \Psi_n\right)\left[\overline{\Psi}_e i\gamma_\lambda\gamma_5 (1 + \gamma_5)\Psi_\nu\right] \tag{1.25}$$

or

$$\mathcal{H} = \frac{G_V}{\sqrt{2}} \left[\overline{\Psi}_p \gamma_\lambda \left(1 - \frac{G_A}{G_V}\gamma_5\right)\Psi_n\right]\left[\overline{\Psi}_e \gamma_\lambda (1 + \gamma_5)\Psi_\nu\right] \tag{1.26}$$

where we have used the relation

$$\gamma_5(1 + \gamma_5) = 1 + \gamma_5$$

Absolute Magnitude Of The Ratio Of V and A Couplings

The transition rate for β decay depends, inter alia, on the phase space available for the leptons and on the effect of the nuclear Coulomb field on the wave function of the emitted electron. The energy available and the Coulomb effect are not directly related to the details of the nuclear states, and it is convenient to separate out these effects in order to exhibit the explicit dependence of the transition rate on the wave functions of the nuclear states involved. This may be accomplished by characterizing the decay, not by its half-life t, but by the so-called comparative half-life ft, where f includes the effects of the energy and the Coulomb field, and is independent of the nuclear states or the beta decay interaction. The quantity f, which is essentially a phase space integral, is defined by the relation

$$\lambda \equiv \frac{\ln 2}{t} \sim \left[G_V^2 |M_F|^2 + G_A^2 |M_{GT}|^2 \right] f \tag{1.27}$$

from which it follows that two decays with the same nuclear structure but with different energies and different nuclear charges will have different half-lives t but should have the same ft value.

Careful measurements of ft values have been made for many nuclei, the most important being those for the neutron and for pure Fermi transitions. The beta decay of the neutron is a mixed (Fermi plus Gamow-Teller) transition, for which

$$|M_F|^2 = 1 \; ; \; |M_{GT}|^2 = 3$$

Thus

$$(ft)_n^{-1} \sim G_V^2 + 3G_A^2 .$$

The measurements of Sosnovskii et al.[24] on neutron decay have yielded the value

$$(ft)_n = 1180 \pm 35 \text{ sec.}$$

On the other hand, ^{14}O decay is a pure Fermi transition. The ^{14}O nucleus is understood as having two protons outside an inert core, and either of them may transform to yield the daughter, $^{14}N^*$. Accordingly, we expect

$$(ft)_0^{-1} \sim 2\, G_V^2$$

with the same constant of proportionality as for neutron decay. The ft value for ^{14}O decay has been measured by Bardin et al.[25] to be

$$(ft)_0 = 3066 \pm 10 \text{ sec.}$$

Thus we get

$$\frac{(ft)_0}{(ft)_n} = \frac{G_V^2 + 3\, G_A^2}{2\, G_V^2} = \frac{3066 \pm 10}{1180 \pm 35}$$

whence $$\frac{G_A^2}{G_V^2} = 1.40 \pm 0.06$$

or $$\left| \frac{G_A}{G_V} \right| = 1.18 \pm 0.03$$

The ft values quoted above have not been corrected for nuclear form factors or radiative effects. A more accurate value

$$\left| \frac{G_A}{G_V} \right| = 1.2 \pm 0.05$$

has been obtained by Blin-Stoyle[26].

Relative Sign Of The V and A Interactions

The relative sign of the vector and axial-vector coupling constants may be obtained from the measurement of the electron and neutrino asymmetries in polarized neutron decay. The angular distribution of the leptons with respect to the direction of the neutron spin may be written in the form[20]

$$1 + B \frac{v}{c} \cos \theta .$$

Theoretical estimates of the asymmetry parameter B are made for both cases $G_A/G_V = \pm 1.2$, and the results are displayed in Table 1. The measurements of the parameters B_e and $B_{\bar{\nu}}$ were made by Burgy et al.[27], who obtained the values

$$B_e = -0.11 \pm 0.02$$

$$B_\nu = +0.88 \pm 0.15$$

These results show clearly that the V and A couplings are of opposite sign :

$$\frac{G_A}{G_V} = -1.2$$

and this establishes the V − A (strictly speaking, V − 1.2 A) character of the beta decay interaction.

The Current-current Form Of The Weak Interaction

In the description of weak processes certain quantities known as currents play an important role. In order to introduce these objects, we consider the process of muon decay. There is considerable evidence that the four-fermion interaction, not only in β decay but in μ decay as well, is of the V − A form. In muon decay the vector and axial-vector coupling constants are equal in magnitude but opposite in sign (so that the interaction is strictly V − A), and the process is described by an interaction

Table 1. Electron and Neutrino Asymmetry Parameters in Polarized Neutron Decay

$\frac{G_A}{G_V}$	B_e	B_ν
+ 1.2	− 1	+ 0.1
− 1.2	− 0.1	+ 1

$$\mathcal{H}_{\mu} = \frac{G}{\sqrt{2}}\left[\overline{\Psi}_{\nu_{\mu}}\gamma_{\lambda}(1 + \gamma_5)\Psi_{\mu}\right]\left[\overline{\Psi}_{e}\gamma_{\lambda}(1 + \gamma_5)\Psi_{\nu_e}\right] \tag{1.28}$$

where G is the muon vector coupling constant. Equation (1.28) may be written in the form

$$\mathcal{H}_{\mu} = \frac{G}{\sqrt{2}} L_{\lambda}(\mu, \nu_{\mu})\, L_{\lambda}(e, \nu_e)$$

where, by analogy with the electromagnetic current, $L_{\lambda}(\mu, \nu_{\mu})$ and $L_{\lambda}(e, \nu_e)$ are known as the leptonic weak currents of the muon and electron, respectively. A similar description is possible for all weak interactions involving leptons. Thus the beta decay interaction

$$\mathcal{H}_{\beta} = \frac{G_V}{\sqrt{2}}\left[\overline{\Psi}_{p}\gamma_{\lambda}\left(1 - \frac{G_A}{G_V}\gamma_5\right)\Psi_{n}\right]\left[\overline{\Psi}_{e}\gamma_{\lambda}(1 + \gamma_5)\Psi_{\nu}\right] \tag{1.26}$$

may be written in the form

$$\mathcal{H}_{\beta} = \frac{G}{\sqrt{2}}\left[\overline{\Psi}_{p}\gamma_{\lambda}\left(g_V + g_A\gamma_5\right)\Psi_{n}\right]\left[\overline{\Psi}_{e}\gamma_{\lambda}(1 + \gamma_5)\Psi_{\nu}\right] \tag{1.29}$$

$$g_V \equiv \frac{G_V}{G} \; ; \; g_A \equiv -\frac{G_A}{G} \tag{1.30}$$

Thus
$$\mathcal{H}_{\beta} = \frac{G}{\sqrt{2}} J_{\lambda}\, L_{\lambda} \tag{1.31}$$

where
$$J_{\lambda} = V_{\lambda} + A_{\lambda} = \overline{\Psi}_{p}\gamma_{\lambda}\left(g_V + g_A\gamma_5\right)\Psi_{n} \tag{1.32}$$

is the nucleon current, and

$$L_{\lambda} = \overline{\Psi}_{e}\gamma_{\lambda}(1 + \gamma_5)\Psi_{\nu} \tag{1.33}$$

is the lepton current. The currents

$$V_\lambda = g_V \overline{\Psi}_p \gamma_\lambda \Psi_n$$

$$A_\lambda = g_A \overline{\Psi}_p \gamma_\lambda \gamma_5 \Psi_n$$

are, respectively, the vector and axial-vector parts of the nucleon current.

It is customary to write the current-current form of the β decay Hamiltonian in terms of the muon vector coupling constant G, which is found, from experiments on the $0^+ \to 0^+$ transition in ^{14}O decay, to be very nearly (within 1 or 2 %) equal to the vector coupling constant G_V in β decay :

$$g_V \equiv \frac{G_V}{G} \approx 1.$$

Unfortunately, the explanation of this remarkably good agreement lies outside the scope of this discussion.

<u>Non-relativistic Approximation and Allowed Selection Rules</u>

Using the interaction (1.29), the transition amplitude for single beta decay may be written in the form

$$\langle f|\mathcal{H}|i\rangle = \frac{G}{\sqrt{2}} \left\{L_4 \left[g_V\int 1 + g_A\int\gamma_5\right] + i\,\vec{L}\cdot\left[g_A\int\vec{\Sigma} - g_V\int\vec{\alpha}\right]\right\} \tag{1.34}$$

where $L_\lambda \equiv (\vec{L}, L_4)$ is the leptonic weak current. We have used the relation

$$-\vec{\alpha}\,\gamma_5 = \vec{\Sigma}$$

and the usual abbreviations, such as

$$\int 1 \equiv \langle f|\sum_n \tau_n^{(+)}|i\rangle$$

$$\int \vec{\alpha} \equiv \langle f|\sum_n \tau_n^{(+)}\vec{\alpha}|i\rangle$$

$\tau_n^{(+)}$ being the (isospin) charge raising operator acting on the n^{th} nucleon.

As we have stated before, the constituents of nuclei are adequately described by the non-relativistic approximation, in which the velocity operator $\vec{\alpha}$ and the operator γ_5 are of the order of magnitude $\frac{v}{c}$, and may be neglected. This corresponds to the neglect of the "small" Dirac components of the nucleon spinors, as discussed before. The effect of these approximations is to reduce the β transition amplitude to

$$\langle f|\mathcal{H}|i\rangle = \frac{G}{\sqrt{2}} \left\{ L_4\, g_V \int 1 + i\, \vec{L}\, g_A . \int \vec{\sigma} \right\} \qquad (1.35)$$

where $\int 1$ and $\int \vec{\sigma}$ are the Fermi and Gamow-Teller matrix elements M_F and M_{GT}, respectively. As we shall see, this reduction yields just the allowed selection rules obtained earlier.

The angular momentum selection rules reflect the conservation of angular momentum. The Fermi selection rules follow from the orthogonality of the angular momentum eigenstates

$$\langle I'M'|IM\rangle = \delta_{I,I'}\, \delta_{M,M'}$$

and this orthogonality is not disturbed in the matrix element

$$\int 1 \equiv \langle f| \sum_n \tau_n^{(+)} |i\rangle \sim \delta_{I,I'}\, \delta_{M,M'}$$

and we have the immediate Fermi selection rule

$$\int 1 \neq 0 \text{ only if } \Delta I = 0.$$

The selection rules characteristic of the matrix element $\int \vec{\sigma}$ follow from the Wigner-Eckart theorem. This theorem states that the matrix element of a tensor operator T^j_m combining states of angular momentum IM and I'M' is proportional to a vector addition, or Clebsch-Gordan, coefficient

$$\langle \delta' I' M' | T^j_m | \delta I M \rangle \sim \langle I j, M m | I' M' \rangle$$

where δ and δ' denote whatever additional quantum numbers are needed to specify the states. The Clebsch-Gordan coefficient $\langle I j, M m | I' M' \rangle$ vanishes unless the angular momenta I, j, I' satisfy the "triangle condition", and this constitutes the selection rules.

In the case of the vector matrix element $\int \vec{\sigma}$ the vector $\vec{\sigma}$ is decomposed into its tensor components

$$T^1_0 = \sigma_z$$

$$T^1_{\pm 1} = \mp \frac{1}{\sqrt{2}} (\sigma_x \pm i\, \sigma_y)$$

We then use the Wigner-Eckart theorem in the form

$$\langle \delta' I' M' | T^1_m | \delta I M \rangle \sim \langle I\, 1, M\, m | I' M' \rangle$$

$$m = 0, \pm 1.$$

The Clebsch-Gordan coefficient $\langle I\, 1, M\, m | I' M' \rangle$ vanishes unless $M' = M + m$ and $I' = I$ or $I \pm 1$; or if $I = I' = 0$. Thus we obtain the Gamow-Teller selection rules

$$\Delta I = 0, 1 \; (\text{no } \vec{0} \to \vec{0})$$

<u>Equivalence of Majorana and Weyl Theories of Massless Spin One-half Particles</u>

A Majorana field is a self-charge-conjugate spin $\frac{1}{2}$ field. It may be denoted by

$$\chi(x) \equiv \chi^{(c)}(x) = C\,\tilde{\bar{\chi}}(x) \tag{1.36}$$

where the charge conjugation matrix C is defined by

$$C\,\tilde{\gamma}_\lambda C^{-1} = -\gamma_\lambda \tag{1.37}$$

$$C^\dagger C = 1 \tag{1.38}$$

$$\tilde{C} = -C. \tag{1.39}$$

It follows from (1.37) that

$$C\tilde{\gamma}_5 = \gamma_5 C. \tag{1.40}$$

Now consider the two-component field $\xi(x)$ defined by

$$\xi(x) = \frac{1}{2}(1+\gamma_5)\Psi(x) \tag{1.41}$$

where $\Psi(x)$ is a massless Dirac field. We make the special Pauli-Gursey transformation[28,29]

$$\Psi(x) \rightarrow \Psi'(x) = \frac{1}{\sqrt{2}}\left[\Psi(x) + \gamma_5 C\tilde{\bar{\Psi}}(x)\right]$$

$$= U\,\Psi(x)\,U^{-1} \tag{1.42}$$

where U is a unitary operator. We now introduce a new field $\chi(x)$, defined by

$$\chi(x) = \frac{1}{\sqrt{2}}\left[\Psi(x) + C\,\tilde{\bar{\Psi}}(x)\right] \tag{1.43}$$

so that $\chi(x)$ is a Majorana field. Now from (1.42)

$$\Psi'(x) = \frac{1}{\sqrt{2}}\left[\Psi(x) + \gamma_5 C\tilde{\gamma}_4\Psi^*(x)\right]$$

$$\overline{\Psi}'(x) = \frac{1}{\sqrt{2}}\left[\overline{\Psi}(x) + \widetilde{\Psi}(x)\gamma_4^* C^\dagger\gamma_5\gamma_4\right]$$

$$\widetilde{\overline{\Psi}}'(x) = \frac{1}{\sqrt{2}}\left[\widetilde{\overline{\Psi}}(x) + \widetilde{\gamma}_4\widetilde{\gamma}_5 C^*\gamma_4\Psi(x)\right]$$

$$C\,\widetilde{\overline{\Psi}}'(x) = \frac{1}{\sqrt{2}}\left[C\,\widetilde{\overline{\Psi}}(x) - C\,\widetilde{\gamma}_4\widetilde{\gamma}_5 C^{-1}\gamma_4\Psi(x)\right]$$

$$= \frac{1}{\sqrt{2}}\left[C\,\widetilde{\overline{\Psi}}(x) - \gamma_5\Psi(x)\right] \qquad (1.44)$$

Thus
$$\xi'(x) = \tfrac{1}{2}(1+\gamma_5)\Psi'(x)$$

$$= \tfrac{1}{2}(1+\gamma_5)\frac{1}{\sqrt{2}}\left[\Psi(x) + \gamma_5 C\,\widetilde{\overline{\Psi}}(x)\right]$$

$$= \frac{1}{2\sqrt{2}}\left[\Psi(x) + \gamma_5\Psi(x) + \gamma_5 C\,\widetilde{\overline{\Psi}}(x) + C\,\widetilde{\overline{\Psi}}(x)\right]$$

$$= \tfrac{1}{2}(1+\gamma_5)\frac{1}{\sqrt{2}}\left[\Psi(x) + C\,\widetilde{\overline{\Psi}}(x)\right]$$

$$= \tfrac{1}{2}(1+\gamma_5)\chi(x)$$

i.e.
$$U\,\xi(x)\,U^{-1} = \tfrac{1}{2}(1+\gamma_5)\chi(x) \qquad (1.45)$$

Now
$$C\,\widetilde{\overline{\xi}}(x) = \tfrac{1}{2}\,C(1-\widetilde{\gamma}_5)\widetilde{\overline{\Psi}}(x)$$

$$= \tfrac{1}{2}(1-\gamma_5)C\,\widetilde{\overline{\Psi}}(x), \text{ using (1.40)},$$

and from (1.44) we get

$$\tfrac{1}{2}(1-\gamma_5)C\,\widetilde{\overline{\Psi}}'(x) = \tfrac{1}{2}(1-\gamma_5)\frac{1}{\sqrt{2}}\left[C\,\widetilde{\overline{\Psi}}(x) - \gamma_5\Psi(x)\right]$$

$$= \tfrac{1}{2}(1-\gamma_5)\frac{1}{\sqrt{2}}\left[\Psi(x) + C\,\widetilde{\overline{\Psi}}(x)\right]$$

$$= \frac{1}{2}(1 - \gamma_5)\chi(x)$$

i.e. $$U\, C\tilde{\bar{\xi}}(x)\, U^{-1} = \frac{1}{2}(1 - \gamma_5)\chi(x) \tag{1.46}$$

Combining (1.45) and (1.46) we get

$$U\left[\xi(x) + C\,\tilde{\bar{\xi}}(x)\right]U^{-1} = \chi(x) \tag{1.47}$$

from which follows the inverse transformation

$$U^{-1}\,\chi(x)\,U = \xi(x) + C\,\tilde{\bar{\xi}}(x) \tag{1.48}$$

By means of equations (1.45) and (1.48) it can be shown[30] that a theory in which a massless spin $\frac{1}{2}$ particle is described by a two-component (Weyl) field and a theory in which it is described by a Majorana field are unitarily, and therefore physically, equivalent. The equivalence may be illustrated by noting that under the unitary transformation U

$$\frac{1}{2}(1 + \gamma_5)\Psi \rightarrow \frac{1}{2}(1 + \gamma_5)\chi \tag{1.45}$$

$$\frac{1}{2}(1 - \gamma_5)\Psi^{(c)} \rightarrow \frac{1}{2}(1 - \gamma_5)\chi. \tag{1.46}$$

In the two-component Weyl theory $\xi = \frac{1}{2}(1 + \gamma_5)\Psi$ represents a (left-handed) neutrino, while $\xi^{(c)} = \frac{1}{2}(1 - \gamma_5)\Psi^{(c)}$ represents a (right-handed) anti-neutrino. We may equally well choose to call these the spin-down and spin-up states of a Majorana particle.

In view of the above considerations we may, if we wish, suppose that the neutrino field Ψ_ν in the lepton current (1.33) describes a Majorana particle. This will be particularly convenient in discussing the neutrinoless mode of $\beta\beta$ decay. We shall use the Majorana representation of the gamma matrices, in which

$$\gamma_j^* = \tilde{\gamma}_j = \gamma_j \; ; \; \gamma_4^* = \tilde{\gamma}_4 = -\gamma_4 \; ; \; \tilde{\gamma}_5 = -\gamma_5 . \qquad (1.49)$$

The charge conjugation matrix C then satisfies the relations

$$C\gamma_j = -\gamma_j C$$

$$C\gamma_4 = +\gamma_4 C$$

and we may choose $C = -\gamma_4$, which gives

$$\begin{aligned}\Psi_\nu &= \Psi_\nu^{(c)} = C\tilde{\overline{\Psi}} \\ &= -\gamma_4\tilde{\gamma}_4\Psi_\nu^* = \Psi_\nu^* .\end{aligned} \qquad (1.50)$$

The neutrino field operator may then be written as

$$\Psi_\nu(\vec{x}) = \sum_{\vec{k},s} a_s(\vec{k})u_s(\vec{k})e^{i\vec{k}\cdot\vec{x}} + a_s^\dagger(\vec{k})u_s^*(\vec{k})e^{-i\vec{k}\cdot\vec{x}} \qquad (1.51)$$

which clearly satisfies (1.50).

CHAPTER II

THE NO-NEUTRINO PROCESS

As we have indicated in Chapter I, double beta decay will be considered as a second-order effect of the usual weak ($\Delta Q = \pm 1$, $\Delta S = 0$) interaction which gives rise, in first order, to single beta decay. The mechanism of the neutrinoless mode is represented by the second order weak diagrams shown in Figure 1.

The transition matrix element is given by the usual second-order time-dependent perturbation theory :

$$M(A, Z-2 \quad A, Z) = \sum_{\mathfrak{N}} \frac{\langle f|H|\mathfrak{N}\rangle\langle\mathfrak{N}|H|i\rangle}{W_{\mathfrak{N}} - W_i} \qquad (2.2)$$

where $|i\rangle$, $|\mathfrak{N}\rangle$ and $|f\rangle$ are the initial, intermediate and final states and W_i and $W_{\mathfrak{N}}$ are the initial and intermediate energies of the nucleon-lepton system. The initial state consists of the parent nucleus (A, Z − 2) and the final state consists of the daughter nucleus (A, Z) together with the two electrons e_1 and e_2. The intermediate state consists of the intermediate nucleus and the intermediate electron e_1 or e_2 plus the intermediate neutrino, each of these leptons with definite momentum and spin orientation. Thus the intermediate energy is

$$W_{\mathfrak{N}} = E_{1 \text{ or } 2} + E_N + E_\nu$$

where E_1 and E_2 are the energies of the electrons 1 and 2, E_N is the

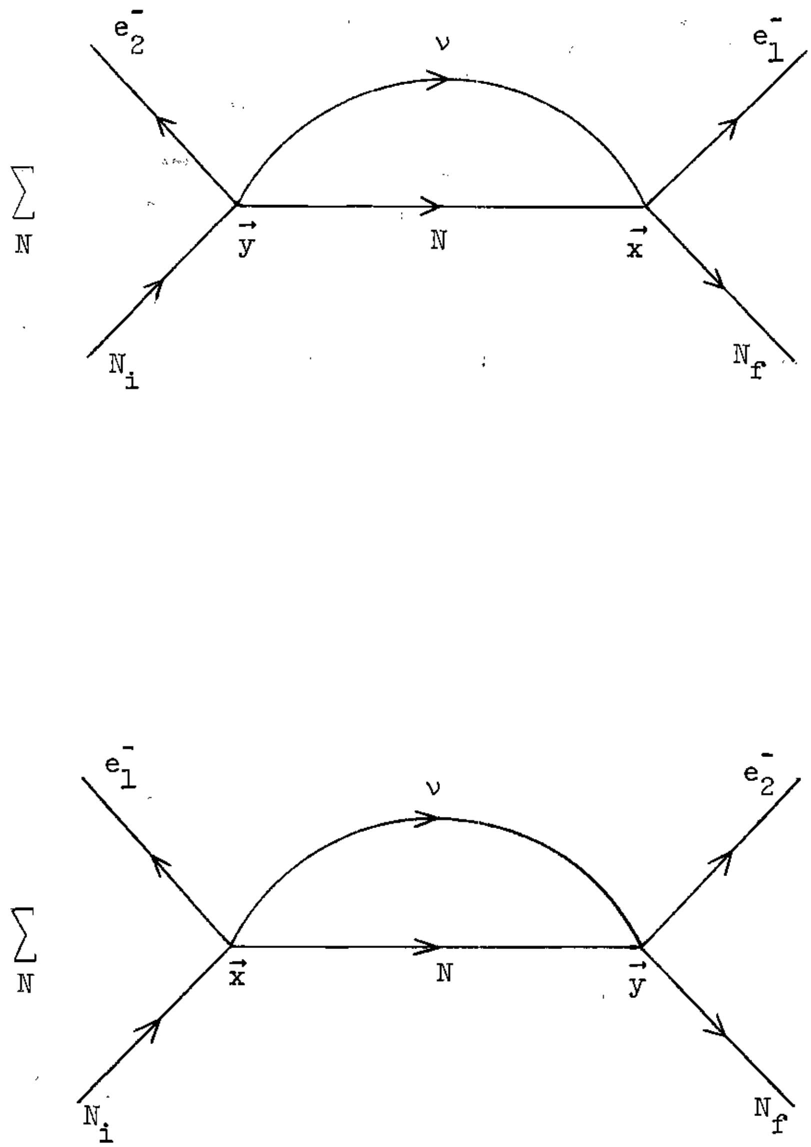

Figure 1. Second Order Weak Diagrams for Neutrinoless ββ Decay

energy of the intermediate nucleus and E_ν is the neutrino energy. The initial energy W_i is just the energy E_i of the parent nucleus. The current-current weak Hamiltonian density is given by

$$\mathcal{H} = \frac{G}{\sqrt{2}} \left[J_\lambda L_\lambda + \text{h.c.} \right] ,$$

J_λ and L_λ being the nucleon and lepton currents respectively. The matrix element for the process is

$$M = \left(1 - P_{12}\right) M' \tag{2.3}$$

where
$$M' = \frac{G^2}{2} \int\int d^3\vec{x}\, d^3\vec{y} \sum_{\vec{k},s} \sum_N$$

$$\frac{\langle e_1 | L_\lambda(\vec{x}) | \nu \rangle \langle N_f | J_\lambda(\vec{x}) | N \rangle \langle e_2 \ \nu | L_\mu(\vec{y}) | 0 \rangle \langle N | J_\mu(\vec{y}) | N_i \rangle}{E_\nu + E_2 + E_N - E_i} \tag{2.4}$$

where $\vec{k}$, s are the momentum and spin projection of the intermediate neutrino and $\sum_N$ denotes summation over a complete set of intermediate nuclear states $|N\rangle$. $|N_i\rangle$ and $|N_f\rangle$ represent the initial and final nuclear states, respectively. P_{12} is the permutation operator which interchanges the roles of the two electrons e_1 and e_2. Thus the factor $\left(1 - P_{12}\right)$ ensures that the right hand side of equation (2.3) is antisymmetric under interchange of the momenta and spin orientations of the emitted electrons.

Lepton Current In Neutrinoless $\beta\beta$ Decay

Let us now examine whether a violation of lepton number conservation can be accomodated within the framework of the two-component form of the leptonic weak current

$$L_\lambda(\vec{x}) \;=\; \overline{\Psi}_e(\vec{x})\gamma_\lambda(1 + \gamma_5)\Psi_\nu(\vec{x}). \tag{1.33}$$

In summing over the modes of the intermediate neutrino we obtain a spin sum

$$\sum_s \langle e_1|L_\lambda(\vec{x})|\nu\rangle\langle e_2\,\nu|L_\mu(\vec{y})|0\rangle$$

which is proportional to

$$\sum_s \left[\overline{u}(\vec{p}_1)\gamma_\lambda(1 + \gamma_5)u_\nu(\vec{k})\right]\left[\overline{u}(\vec{p}_2)\gamma_\mu(1 + \gamma_5)u_\nu^*(\vec{k})\right].$$

Upon transposition of the second bracket, this becomes

$$\overline{u}(\vec{p}_1)\gamma_\lambda(1 + \gamma_5)\left[\sum_s u_\nu(\vec{k})u_\nu^\dagger(\vec{k})\right](1 - \gamma_5)\widetilde{\gamma}_\mu\widetilde{\gamma}_4\;u^*(\vec{p}_2)$$

We now insert a factor of $\frac{1}{2}\left(1 + \frac{\vec{\alpha}\cdot\vec{k}}{k}\right)$, which is the projection operator for the positive energy one particle states of the neutrino. This gives

$$\overline{u}(\vec{p}_1)\gamma_\lambda(1 + \gamma_5)\frac{1}{2}\left(1 + \frac{\vec{\alpha}\cdot\vec{k}}{k}\right)\left[\sum_s u_\nu u_\nu^\dagger + v_\nu v_\nu^\dagger\right](1 - \gamma_5)\gamma_4\gamma_\mu\;u^*(\vec{p}_2)$$

where v_ν is the negative energy neutrino spinor amplitude. It follows from (1.49) that $\widetilde{\gamma}_\mu\widetilde{\gamma}_4 \;=\; \gamma_4\gamma_\mu$. If we use the normalization

$$u_r^\dagger u_s \;=\; \delta_{rs}\;;\;\; v_r^\dagger v_s \;=\; -\,\delta_{rs}$$

then

$$\sum_s (u_\nu u_\nu^\dagger + v_\nu v_\nu^\dagger) \;=\; 1$$

and we get

$$\sum_{s} \langle e_1 | L_\lambda(\vec{x}) | \nu \rangle \langle e_2 \ \nu | L_\mu(\vec{y}) | 0 \rangle \ \sim$$

$$\sim \ \bar{u}(\vec{p}_1)\gamma_\lambda(1 + \gamma_5)\frac{1}{2}\left(1 + \frac{\vec{\alpha}\cdot\vec{k}}{k}\right)(1 - \gamma_5)\gamma_4\gamma_\mu \ u^*(\vec{p}_2) \ =$$

$$= \ \bar{u}(\vec{p}_1)\gamma_\lambda \ \frac{1}{2}\left(1 + \frac{\vec{\alpha}\cdot\vec{k}}{k}\right)\gamma_4\gamma_\mu(1 + \gamma_5)(1 - \gamma_5)u^*(\vec{p}_2) \ \equiv \ 0$$

since γ_5 anticommutes with γ_μ and commutes with $\vec{\alpha}$, and $(1 + \gamma_5)(1 - \gamma_5)$ is identically equal to zero. We see therefore that if the lepton weak current is of the form (1.33) neutrinoless $\beta\beta$ decay cannot take place : it is clearly defeated by the opposite helicities of the emitted and absorbed particles. To ensure that the virtual neutrino which is emmitted is subsequently absorbed, the lepton current must be modified to include a term with the helicity projection operator $(1 - \gamma_5)$, weighted by a factor η which determines the extent of the lepton number violation :

$$L_\lambda(\vec{x}) \ = \ \bar{\Psi}_e(\vec{x})\gamma_\lambda\left[(1 + \gamma_5) + \eta(1 - \gamma_5)\right]\Psi_\nu(\vec{x}) \qquad (2.5)$$

<u>Calculation of the Matrix Element</u>

With this modification (2.4) becomes

$$M' \ = \ \frac{G^2}{2}\int\int d^3\vec{x} \ d^3\vec{y} \sum_{\vec{k},s}\sum_{N} e^{-i(\vec{p}_1\cdot\vec{x} + \vec{p}_2\cdot\vec{y})} \ e^{-i\vec{k}\cdot(\vec{y} - \vec{x})} \qquad \times$$

$$\left\{\bar{u}(\vec{p}_1)\gamma_\lambda\left[(1 + \gamma_5) + \eta(1 - \gamma_5)\right]u_\nu(\vec{k})\right\}\left\{\bar{u}(\vec{p}_2)\gamma_\mu\left[(1 + \gamma_5) + \eta(1 - \gamma_5)\right]u_\nu^*(\vec{k})\right\} \times$$

$$\frac{\langle N_f | J_\lambda(\vec{x}) | N \rangle \langle N | J_\mu(\vec{y}) | N_i \rangle}{k + E_2 + E_N - E_i} \qquad (2.6)$$

Using the same procedure as before (transposition of the second curly bracket) the product of the two curly brackets can be shown to be

$$\bar{u}(\vec{p}_1)\gamma_\lambda \tfrac{1}{2}\left(1 + \frac{\vec{\alpha}\cdot\vec{k}}{k}\right)\gamma_4\gamma_\mu\left[(1+\gamma_5) + \eta(1-\gamma_5)\right]\left[(1-\gamma_5) + \eta(1+\gamma_5)\right] u^*(\vec{p}_2)$$

$$= (2\eta)\ \bar{u}(\vec{p}_1)\gamma_\lambda \left(1 + \frac{\vec{\alpha}\cdot\vec{k}}{k}\right)\gamma_4\gamma_\mu\ u^*(\vec{p}_2).$$

We shall now replace the energy E_N of the intermediate nucleus by an average value $\langle E_N\rangle$, as a result of which a closure summation may be performed. This yields

$$M' = \frac{G^2}{2}\int\int d^3\vec{x}\ d^3\vec{y} \sum_{\vec{k}} \left(\ell_{\lambda\mu} + i\ \vec{\ell}'_{\lambda\mu}\cdot\frac{\vec{k}}{k}\right) e^{-i(\vec{p}_1\cdot\vec{x} + \vec{p}_2\cdot\vec{y})} \times$$

$$\frac{e^{-i\vec{k}\cdot(\vec{y}-\vec{x})}}{E_2 + k + \langle E_N\rangle - E_i} \langle N_f|J_\lambda(\vec{x})\ J_\mu(\vec{y})|N_i\rangle \tag{2.7}$$

where

$$\ell_{\lambda\mu} \equiv (2\eta)\ \bar{u}(\vec{p}_1)\gamma_\lambda\gamma_4\gamma_\mu\ u^*(\vec{p}_2) \tag{2.8}$$

$$\vec{\ell}'_{\lambda\mu} \equiv -(2\eta)\ \bar{u}(\vec{p}_1)\gamma_\lambda\vec{\gamma}\ \gamma_\mu\ u^*(\vec{p}_2) \tag{2.9}$$

The second term in equation (2.3) is obtained from

$$P_{12}\, M' = \frac{G^2}{2}\int\int d^3\vec{x}\ d^3\vec{y} \sum_{\vec{k},s}\sum_{N}$$

$$\frac{\langle e_2|L_\mu(\vec{y})|\nu\rangle\langle N_f|J_\mu(\vec{y})|N\rangle\langle e_1\ \nu|L_\lambda(\vec{x})|0\rangle\langle N|J_\lambda(\vec{x})|N_i\rangle}{E_1 + k + E_N - E_i} \tag{2.10}$$

or $$P_{12}\, M' = \frac{G^2}{2} \int\int d^3\vec{x}\; d^3\vec{y} \sum_{\vec{k}} \left(\ell_{\lambda\mu} + i\; \vec{\ell}\,'_{\lambda\mu} \cdot \frac{\vec{k}}{k}\right) e^{-i(\vec{p}_1\cdot\vec{x} + \vec{p}_2\cdot\vec{y})} \times$$

$$\frac{e^{-i\vec{k}\cdot(\vec{x} - \vec{y})}}{E_1 + k + \langle E_N\rangle - E_i} \langle N_f | J_\mu(\vec{y})\; J_\lambda(\vec{x}) | N_i \rangle . \qquad (2.11)$$

Since the equal-time nucleon current operators $J_\lambda(\vec{x})$ and $J_\mu(\vec{y})$ commute, the total matrix element is given by

$$M = (1 - P_{12})M' = \frac{G^2}{2} \int\int d^3\vec{x}\; d^3\vec{y} \int \frac{d^3\vec{k}}{(2\pi)^3} \left(\ell_{\lambda\mu} + i\; \vec{\ell}\,'_{\lambda\mu} \cdot \frac{\vec{k}}{k}\right) \times \qquad (2.12)$$

$$e^{-i(\vec{p}_1\cdot\vec{x} + \vec{p}_2\cdot\vec{y})} \left[\frac{e^{-i\vec{k}\cdot(\vec{y} - \vec{x})}}{E_2 + k + \delta} - \frac{e^{i\vec{k}\cdot(\vec{y} - \vec{x})}}{E_1 + k + \delta} \right] \langle N_f | J_\lambda(\vec{x})\; J_\mu(\vec{y}) | N_i \rangle$$

where $$\delta \equiv \langle E_N\rangle - E_i \qquad (2.13)$$

and we have made the usual replacement $\sum_{\vec{k}} \rightarrow \int \frac{d^3\vec{k}}{(2\pi)^3}$.

There is no upper limit to the energy k of the intermediate neutrino. However, contributions from very high values of k are averaged out to zero by the large number of oscillations of the exponential terms within a region in which the nuclear matrix element may be considered constant. The main contribution to the k integration comes from the region in which $k\, R \approx \hbar$, where R is the average separation of nucleons inside the nucleus (this is taken to be the nuclear radius). Thus the most effective virtual neutrinos have an energy

$$E_\nu = kc \approx \frac{\hbar c}{R} = \frac{2 \times 10^{-11}\ \text{MeV-cm}}{1.2 \times A^{1/3} \times 10^{-13}\ \text{cm}}$$

$$\approx 30 \text{ MeV for nuclei with } A \approx 130.$$

The energy release for the nuclei of interest is less than 5 MeV, and it has been estimated[12,36] that δ is no more than 10 MeV, and probably a lot less. Because of the relatively large value of k we shall make a Taylor expansion of $(k + E + \delta)^{-1}$ in powers of $\frac{E + k}{k}$, keeping only the linear term :

$$\frac{1}{k + E + \delta} \approx \frac{1}{k}\left(1 - \frac{E + \delta}{k}\right). \tag{2.14}$$

Inserted into equation (2.12), this gives

$$M = \frac{G^2}{2} \int\int\int d^3\vec{x}\, d^3\vec{y}\, \frac{d^3\vec{k}}{(2\pi)^3} \left(\ell_{\lambda\mu} + i\, \vec{\ell}\,'_{\lambda\mu}\cdot\frac{\vec{k}}{k}\right) e^{-i(\vec{p}_1\cdot\vec{x} + \vec{p}_2\cdot\vec{y})}$$

$$\left[\left(\frac{1}{k} - \frac{E_2 + \delta}{k^2}\right) e^{-i\vec{k}\cdot(\vec{y} - \vec{x})} - \left(\frac{1}{k} - \frac{E_1 + \delta}{k^2}\right) e^{i\vec{k}\cdot(\vec{y} - \vec{x})}\right] \langle N_f | J_\lambda(\vec{x}) J_\mu(\vec{y}) | N_i \rangle \tag{2.15}$$

We shall now perform the $\vec{k}$ integration. The following integrals (See Appendix I) will be needed for the evaluation :

$$\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{e^{-i\,\vec{k}\cdot\vec{r}}}{k^2} = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\,\vec{k}\cdot\vec{r}}}{k^2} = \frac{1}{4\pi r} \tag{2.16}$$

$$\vec{\ell}\,'_{\lambda\mu}\cdot\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{\vec{k}}{k^2} e^{-i\,\vec{k}\cdot\vec{r}} = -\vec{\ell}\,'_{\lambda\mu}\cdot\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{\vec{k}}{k^2} e^{i\,\vec{k}\cdot\vec{r}} = \frac{1}{4\pi i} \frac{\vec{\ell}\,'_{\lambda\mu}\cdot\vec{r}}{r^3} \tag{2.17}$$

$$\vec{\ell}'_{\lambda\mu}\cdot\int\frac{d^3\vec{k}}{(2\pi)^3}\,\frac{\vec{k}}{k^3}\,e^{-i\,\vec{k}\cdot\vec{r}} = -\,\vec{\ell}'_{\lambda\mu}\cdot\int\frac{d^3\vec{k}}{(2\pi)^3}\,\frac{\vec{k}}{k^3}\,e^{i\,\vec{k}\cdot\vec{r}} = \frac{1}{2\pi^2 i}\,\frac{\vec{\ell}'_{\lambda\mu}\cdot\vec{r}}{r^2} \qquad (2.18)$$

where $\vec{r} \equiv \vec{y} - \vec{x}.$

On carrying out the $\vec{k}$ integration, we get

$$M = \frac{G^2}{8\pi}\,(E_1 - E_2)\,\ell_{\lambda\mu}\,\langle N_f|\int\int d^3\vec{x}\,d^3\vec{y}\;J_\lambda(\vec{x})J_\mu(\vec{y})\;\frac{e^{-i(\vec{p}_1\cdot\vec{x} + \vec{p}_2\cdot\vec{y})}}{|\vec{y}-\vec{x}|}\;|N_i\rangle\; +$$

$$+\,\frac{G^2}{8\pi}\,\vec{\ell}'_{\lambda\mu}\cdot\langle N_f|2\int\int d^3\vec{x}\,d^3\vec{y}\;J_\lambda(\vec{x})J_\mu(\vec{y})\;\frac{e^{-i(\vec{p}_1\cdot\vec{x} + \vec{p}_2\cdot\vec{y})}}{|\vec{y}-\vec{x}|^3}\;(\vec{y}-\vec{x})\,|N_i\rangle\; - \qquad (2.20)$$

$$-\,\frac{G^2}{8\pi^2}\,(E_1 + E_2 + 2\delta)\,\vec{\ell}'_{\lambda\mu}\cdot\langle N_f|2\int\int d^3\vec{x}\,d^3\vec{y}\;J_\lambda(\vec{x})J_\mu(\vec{y})\;\frac{e^{-i(\vec{p}_1\cdot\vec{x} + \vec{p}_2\cdot\vec{y})}}{|\vec{y}-\vec{x}|^2}\;(\vec{y}-\vec{x})\,|N_i\rangle$$

<u>The Nuclear Matrix Elements</u>

The constituents of nuclei are adequately described by the non-relativistic approximation, nuclear energies being much smaller than nucleon rest masses (MeV vs GeV). The nucleon current will be written as a sum of terms, each of which acts only on one nucleon :

$$J_\lambda(\vec{x}) = \sum_n \tau_n^{(+)}\gamma_4\left[\gamma_\lambda(g_V + g_A\gamma_5)\right]\delta^3(\vec{x}-\vec{r}_n) \qquad (2.21)$$

$$= \sum_n \tau_n^{(+)}\;(\Gamma_\lambda)_n\;\delta^3(\vec{x}-\vec{r}_n)$$

where $\vec{r}_n$ is the position vector of the n^{th} nucleon and $\tau_n^{(+)}$ is the (isotopic) charge raising operator. If the n^{th} nucleon in the nuclear wave function happens to be a neutron, $\tau_n^{(+)}$ converts it into a proton ; while if it is a proton, the result is zero. From above, we have

$$(\Gamma_4)_n = g_V + g_A \gamma_5 \tag{2.22}$$

$$(\vec{\Gamma})_n = g_V \gamma_4 \vec{\gamma} + g_A \gamma_4 \vec{\gamma}\, \gamma_5$$

$$= - i\, g_V\, \vec{\alpha} + i\, g_A\, \vec{\Sigma}\,. \tag{2.23}$$

In the non-relativistic approximation the operators γ_5 and $\vec{\alpha}$ are discarded, which is equivalent to neglecting the 'small' Dirac spinor components of the nuclear states. In this approximation, we therefore have

$$(\Gamma_\lambda)_n = g_V\, \delta_{\lambda 4} + i\, g_A (\sigma_\lambda)_n (1 - \delta_{\lambda 4}), \tag{2.24}$$

where σ_j are the 2 x 2 Pauli spin matrices. We shall also write

$$J_\mu(\vec{y}) = \sum_m \tau_m^{(+)} (\Gamma_\mu)_m\, \delta^3(\vec{y} - \vec{r}_m)\,. \tag{2.25}$$

On performing the $\vec{x}$ and $\vec{y}$ integration, we obtain from (2.20)

$$M = \frac{G^2}{8\pi} (E_1 - E_2) \ell_{\lambda\mu} \langle N_f | \sum_{n,m} \tau_n^{(+)} \tau_m^{(+)} (\Gamma_\lambda)_n (\Gamma_\mu)_m \frac{e^{-i(\vec{p}_1 \cdot \vec{r}_n + \vec{p}_2 \cdot \vec{r}_m)}}{|\vec{r}_n - \vec{r}_m|} |N_i\rangle -$$

$$- \frac{G^2}{8\pi} \vec{\ell}\,'_{\lambda\mu} \cdot \langle N_f | 2 \sum_{n,m} \tau_n^{(+)} \tau_m^{(+)} (\Gamma_\lambda)_n (\Gamma_\mu)_m \frac{e^{-i(\vec{p}_1 \cdot \vec{r}_n + \vec{p}_2 \cdot \vec{r}_m)}}{|\vec{r}_n - \vec{r}_m|^3} (\vec{r}_n - \vec{r}_m) |N_i\rangle \quad +$$

$$+\frac{G^2}{8\pi^2}(E_1+E_2+2\delta)\vec{\ell}^{\,\prime}_{\lambda\dot{\mu}}\;\langle N_f|2\sum_{n,m}\tau_n^{(+)}\tau_m^{(+)}(\Gamma_\lambda)_n(\Gamma_\mu)_m\frac{e^{-i(\vec{p}_1\cdot\vec{r}_n+\vec{p}_2\cdot\vec{r}_m)}}{|\vec{r}_n-\vec{r}_m|^2}(\vec{r}_n-\vec{r}_m)|N_i\rangle \tag{2.26}$$

where
$$(\Gamma_\lambda)_n(\Gamma_\mu)_m = g_V^2\,\delta_{\lambda 4}\delta_{\mu 4} - g_A^2\,(\sigma_\lambda)_n(\sigma_\mu)_m(1-\delta_{\lambda 4})(1-\delta_{\mu 4}) +$$
$$+\,i\,g_V g_A\left[\delta_{\lambda 4}(\sigma_\mu)_m(1-\delta_{\mu 4}) + \delta_{\mu 4}(\sigma_\lambda)_n(1-\delta_{\lambda 4})\right] \tag{2.27}$$

As indicated before, in all the double beta decays of interest, the nuclear states of the initial and final nuclei have spin-parity 0^+. This means that we must extract from the operators sandwiched between the nuclear states only those parts which are invariant under rotations and inversions of the nuclear coordinates. When this is done (the details of the calculation are reserved for Appendix II), we obtain

$$M = \frac{G^2}{8\pi}(E_1-E_2)(2\pi)\left(\bar{u}\,\gamma_4\,u^*\right)\Big\{g_V^2\,\langle N_f|\sum_{n,m}\tau_n^{(+)}\tau_m^{(+)}\frac{1}{r_{nm}}|N_i\rangle +$$

$$+\,g_A^2\,\langle N_f|\sum_{n,m}\tau_n^{(+)}\tau_m^{(+)}\frac{\vec{\sigma}_n\cdot\vec{\sigma}_m}{r_{nm}}|N_i\rangle\Big\} +$$

$$+\frac{1}{3}\frac{G^2}{8\pi}(2\pi)\left(\bar{u}\,i\vec{\gamma}\cdot\vec{p}\,u^*\right)\Big\{g_V^2\,\langle N_f|\sum_{n,m}\tau_n^{(+)}\tau_m^{(+)}\frac{1}{r_{nm}}|N_i\rangle -$$

$$-\,g_A^2\langle N_f|\sum_{n,m}\tau_n^{(+)}\tau_m^{(+)}\frac{\vec{\sigma}_n\cdot\vec{\sigma}_m}{r_{nm}}|N_i\rangle + 2g_A^2\langle N_f|\sum_{n,m}\tau_n^{(+)}\tau_m^{(+)}\frac{(\vec{\sigma}_n\cdot\vec{r}_{nm})(\vec{\sigma}_m\cdot\vec{r}_{nm})}{r_{nm}^3}|N_i\rangle\Big\} -$$

$$-\frac{1}{3}\frac{G^2}{8\pi^2}(E_1+E_2+2\delta)(2\eta)\left(\overline{u}\, i\, \vec{\gamma}\cdot\vec{p}\, u^*\right)\left\{g_V^2\, \langle N_f|\sum_{n,m}\tau_n^{(+)}\tau_m^{(+)}|N_i\rangle\right. -$$

$$\left. - g_A^2\,\langle N_f|\sum_{n,m}\tau_n^{(+)}\tau_m^{(+)}\vec{\sigma}_n\cdot\vec{\sigma}_m|N_i\rangle + 2g_A^2\,\langle N_f|\sum_{n,m}\tau_n^{(+)}\tau_m^{(+)}\,\frac{(\vec{\sigma}_n\cdot\vec{r}_{nm})(\vec{\sigma}_m\cdot\vec{r}_{nm})}{r_{nm}^2}|N_i\rangle\right\} \tag{2.28}$$

where
$$\vec{p} \equiv \vec{p}_1 - \vec{p}_2 \tag{2.29}$$

$$r_{nm} \equiv |\vec{r}_n - \vec{r}_m| \tag{2.30}$$

If the factor $(r_{nm})^{-1}$ is removed from the integrals as being of the order of magnitude of R^{-1}, where R is the nuclear radius, we get

$$M = \frac{G^2}{4\pi}\eta(E_1-E_2)\left[\overline{u}(\vec{p}_1)\gamma_4 u^*(\vec{p}_2)\right]\frac{1}{R}\left\{g_V^2\, M_1 + g_A^2\, M_2\right\} +$$

$$+\frac{1}{3}\frac{G^2}{4\pi}\eta\left[\overline{u}(\vec{p}_1) i\vec{\gamma}\cdot\vec{p}\, u^*(\vec{p}_2)\right]\frac{1}{R}\left\{g_V^2\, M_1 - g_A^2(M_2-2M_3)\right\} - \tag{2.31}$$

$$-\frac{1}{3}\frac{G^2}{4\pi^2}\eta(E_1+E_2+2\delta)\left[\overline{u}(\vec{p}_1) i\vec{\gamma}\cdot\vec{p}\, u^*(\vec{p}_2)\right]\left\{g_V^2\, M_1 - g_A^2(M_2-2M_3)\right\}$$

where
$$M_1 \equiv \langle N_f|\sum_{n,m}\tau_n^{(+)}\tau_m^{(+)}|N_i\rangle \tag{2.32}$$

$$M_2 \equiv \langle N_f|\sum_{n,m}\tau_n^{(+)}\tau_m^{(+)}\,\vec{\sigma}_n\cdot\vec{\sigma}_m|N_i\rangle \tag{2.33}$$

$$M_3 \equiv \langle N_f|\sum_{n,m}\tau_n^{(+)}\tau_m^{(+)}(\vec{\sigma}_n\cdot\hat{r}_{nm})(\vec{\sigma}_m\cdot\hat{r}_{nm})|N_i\rangle \tag{2.34}$$

We shall make the assumption that pairs of like nucleons in the nucleus are predominantly in singlet spin states[31]. The singlet two-nucleon spin state may be written as

$$|0\ 0\rangle_{n,m} = \frac{1}{\sqrt{2}}\,|n(\tfrac{1}{2})m(-\tfrac{1}{2}) - n(-\tfrac{1}{2})m(\tfrac{1}{2})\,\rangle \tag{2.35}$$

Thus $\langle 0\ 0|0\ 0\rangle = \frac{1}{2}\,\langle n(\frac{1}{2})m(-\frac{1}{2}) - n(-\frac{1}{2})m(\frac{1}{2})\,|\,n(\frac{1}{2})m(-\frac{1}{2}) - n(-\frac{1}{2})m(\frac{1}{2})\,\rangle$

$$= 1 \tag{2.36}$$

Now $\vec{\sigma}_n \cdot \vec{\sigma}_m = \sum_a (-)^a\, \sigma_n^{(a)} \sigma_m^{(-a)}\ ; \quad a = +,\ 0,\ -$

$$= -\,\sigma_n^{(+)}\sigma_m^{(-)} + \sigma_n^{(0)}\sigma_m^{(0)} - \sigma_n^{(-)}\sigma_m^{(+)} \tag{2.37}$$

where $$\sigma^{(\pm)} = \mp\frac{1}{\sqrt{2}}\,(\sigma_x \pm i\sigma_y) \tag{2.38}$$

$$\sigma^{(0)} = \sigma_z \tag{2.39}$$

Thus $\langle 0\ 0|\vec{\sigma}_n \cdot \vec{\sigma}_m|0\ 0\rangle =$

$$= \frac{1}{2}\,\langle n(\tfrac{1}{2})m(-\tfrac{1}{2}) - n(-\tfrac{1}{2})m(\tfrac{1}{2})\,|\Big(-\,\sigma_n^{+}\sigma^{-} + \sigma_n^{0}\sigma_m^{0} - \sigma_n^{-}\sigma_m^{+}\Big)|\,n(\tfrac{1}{2})m(-\tfrac{1}{2}) - n(-\tfrac{1}{2})m(\tfrac{1}{2})\,\rangle \tag{2.40}$$

Remembering that

$$\vec{s} = \frac{1}{2}\,\vec{\sigma}$$

$$s^{\pm}|j,\mu\rangle = \sqrt{j(j+1) - \mu(\mu+1)}\ |j,\mu \pm 1\,\rangle$$

$$s_z|j,\mu\rangle = \mu\ |j,\mu\rangle$$

we get $$\langle 0\ 0|\vec{\sigma}_n \cdot \vec{\sigma}_m|0\ 0\rangle \ =$$

$$2\ \langle n(\tfrac{1}{2})m(-\tfrac{1}{2}) - n(-\tfrac{1}{2})m(\tfrac{1}{2})|-\frac{1}{2}\ n(\tfrac{1}{2})m(-\tfrac{1}{2}) - \frac{1}{4}\ n(\tfrac{1}{2})m(-\tfrac{1}{2}) - \frac{1}{4}\ n(-\tfrac{1}{2})m(\tfrac{1}{2}) + \frac{1}{2}\ n(-\tfrac{1}{2})m(\tfrac{1}{2})\rangle$$

$$= -\frac{3}{2}\ \langle n(\tfrac{1}{2})m(-\tfrac{1}{2}) - n(-\tfrac{1}{2})m(\tfrac{1}{2})|n(\tfrac{1}{2})m(-\tfrac{1}{2}) - n(-\tfrac{1}{2})m(\tfrac{1}{2})\rangle \ = \ -3. \qquad (2.41)$$

Hence, from (2.36) and (2.41), we see that

$$M_1 \ = \ -\frac{1}{3}\,M_2 \qquad (2.42)$$

If we also make the estimate[13,36] that $M_2 = M_3$, the expression (2.31) reduces to

$$M \ = \ \frac{G^2}{4\pi}\,\eta\,\Big\{(E_1 - E_2)\Big[\overline{u}(\vec{p}_1)\gamma_4\ u^*(\vec{p}_2)\Big]\frac{1}{R} + \frac{1}{3}\Big[\overline{u}(\vec{p}_1)i\vec{\gamma}\cdot\vec{p}\ u^*(\vec{p}_2)\Big]\frac{1}{R} \ -$$

$$-\frac{1}{3\pi}\Big(E_1 + E_2 + 2\delta\Big)\Big[\overline{u}(\vec{p}_1)i\vec{\gamma}\cdot\vec{p}\ u^*(\vec{p}_2)\Big]\Big\}\Big(-\frac{1}{3}\,g_V^2 + g_A^2\Big)\,M_2 \qquad (2.43)$$

The above expression may be somewhat simplified by recalling that the spinor amplitude $u(\vec{p})$ satisfies the relations

$$(m + i\,\not{p})u(\vec{p}) \ = \ 0 \qquad (2.44)$$

$$\overline{u}(\vec{p})\Big[m + i\,\not{p}\Big] \ = \ 0 \qquad (2.45)$$

where $$\not{p} \ \equiv \ \gamma\cdot p$$

Taking the complex conjugate of (2.44), we get

$$\Big[m - i(\vec{\gamma}^*\cdot\vec{p}^* + \gamma_4^* p_4^*)\Big]\,u^*(\vec{p}) \ = \ 0$$

where in the Majorana representation we are using

$$\vec{\gamma}^{\,*} = \vec{\gamma} \;;\; \gamma_4^* = -\gamma_4$$

Further $$\vec{p}^{\,*} = \vec{p} \;;\; p_4^* = -p_4$$

Thus we obtain

$$(m - i\not{p})\, u^*(\vec{p}) = 0 \tag{2.46}$$

From (2.46) and (2.45) we then have

$$\overline{u}(\vec{p}_1)\Big[m - i\not{p}_2\Big]\, u^*(\vec{p}_2) = 0$$

$$\overline{u}(\vec{p}_1)\Big[m + i\not{p}_1\Big]\, u^*(\vec{p}_2) = 0$$

Adding these two equations, we get

$$2m\, \overline{u}(\vec{p}_1)u^*(\vec{p}_2) + \overline{u}(\vec{p}_1)\Big[i(\not{p}_1 - \not{p}_2)\Big]\, u^*(\vec{p}_2) = 0$$

whence

$$\overline{u}(\vec{p}_1)i\vec{\gamma}\cdot\vec{p}\; u^*(\vec{p}_2) = (E_1 - E_2)\overline{u}(\vec{p}_1)\gamma_4 u^*(\vec{p}_2) - 2m\, \overline{u}(\vec{p}_1)u^*(\vec{p}_2) \tag{2.47}$$

$$\vec{p} \equiv \vec{p}_1 - \vec{p}_2$$

Substituting (2.47) into (2.43), we obtain

$$M = \frac{1}{3}\frac{G^2}{4\pi}\,\eta(g_A^2 - \frac{1}{3}\, g_V^2)\Big[(E_1 - E_2)\overline{u}(\vec{p}_1)\gamma_4 u^*(\vec{p}_2)\Big(\frac{4}{R} - \frac{E_1 + E_2 + 2\delta}{\pi}\Big) -$$

$$- 2m\, \overline{u}(\vec{p}_1)u^*(\vec{p}_2)\Big(\frac{1}{R} - \frac{E_1 + E_2 + 2\delta}{\pi}\Big)\Big]\, M_2 \tag{2.48}$$

$$\text{or}\quad M = \frac{1}{3}\frac{G^2}{4\pi}\eta(g_A^2 - \frac{1}{3}g_V^2)\left[A\,\bar{u}(\vec{p}_1)\gamma_4 u^*(\vec{p}_2) - B\,\bar{u}(\vec{p}_1)u^*(\vec{p}_2)\right] M_2 \qquad (2.49)$$

$$\text{where}\quad A = (E_1 - E_2)\left[\frac{4}{R} - \frac{E_1 + E_2 + 2\delta}{\pi}\right] \qquad (2.50)$$

$$B = 2m\left[\frac{1}{R} - \frac{E_1 + E_2 + 2\delta}{\pi}\right] \qquad (2.51)$$

We now take the squared modulus of the matrix element M and sum over the electron spins, to give

$$\sum_{s_1,s_2} |M|^2 = \frac{1}{81}\left(\frac{G^2}{4\pi}\eta\right)^2(3g_A^2 - g_V^2)^2\left[A^2\frac{E_1E_2 - 1 + \vec{p}_1\cdot\vec{p}_2}{E_1E_2} + \right.$$

$$\left. + B^2\frac{E_1E_2 - 1 - \vec{p}_1\cdot\vec{p}_2}{E_1E_2} + 2\,AB\,\frac{E_1 - E_2}{E_1E_2}\right]|M_2|^2 \qquad (2.52)$$

where we have used (see Appendix III)

$$\sum_{s_1,s_2} |\bar{u}(\vec{p}_1)\gamma_4 u^*(\vec{p}_2)|^2 = \frac{E_1E_2 - 1 + \vec{p}_1\cdot\vec{p}_2}{E_1E_2} \qquad (2.53)$$

$$\sum_{s_1,s_2} |\bar{u}(\vec{p}_1)u^*(\vec{p}_2)|^2 = \frac{E_1E_2 - 1 - \vec{p}_1\cdot\vec{p}_2}{E_1E_2} \qquad (2.54)$$

$$\sum_{s_1,s_2}\left[\bar{u}(\vec{p}_1)\gamma_4 u^*(\vec{p}_2)\right]^\dagger\left[\bar{u}(\vec{p}_1)u^*(\vec{p}_2)\right] = \sum_{s_1,s_2}\left[\bar{u}(\vec{p}_1)u^*(\vec{p}_2)\right]^\dagger\left[\bar{u}(\vec{p}_1)\gamma_4 u^*(\vec{p}_2)\right]$$

$$= -\frac{1}{E_1E_2}(E_1 - E_2) \qquad (2.55)$$

Equation (2.52) can be written in the form

$$\sum_{s_1,s_2} |M|^2 = P(E_1,E_2) + Q(E_1,E_2) \cos\theta \tag{2.56}$$

where
$$\cos\theta = \frac{\vec{p}_1 \cdot \vec{p}_2}{p_1\, p_2} \tag{2.57}$$

$$P = \frac{1}{81}\left(\frac{G^2}{4\pi}\eta\right)^2\left(3g_A^2 - g_V^2\right)^2\left[(A^2 + B^2)\frac{E_1E_2 - 1}{E_1E_2} + 2AB\frac{E_1 - E_2}{E_1E_2}\right]|M_2|^2 \tag{2.58}$$

$$Q = \frac{1}{81}\left(\frac{G^2}{4\pi}\eta\right)^2\left(3g_A^2 - g_V^2\right)^2\left(A^2 - B^2\right)\frac{p_1p_2}{E_1E_2}|M_2|^2 \tag{2.59}$$

The calculation of the decay rate will be carried out using units in which $\hbar = c = m = 1$, m being the electron mass. Thus the electron energies are given by $E_i = \epsilon_i + 1$ ($i = 1, 2$), where ϵ_i is the kinetic energy of the i^{th} electron. The energy release ϵ_o is equal to the maximum kinetic energy of either electron, if we neglect the recoil of the daughter nucleus.

Using (2.56), the decay rate $\lambda^{(o)}$ (we shall use the superscripts (o) and (2) to distinguish between the rates of no-neutrino and two-neutrino decay) is given by

$$\lambda^{(o)} = 2\pi\int\frac{d^3\vec{p}_1}{(2\pi)^3}\frac{d^3\vec{p}_2}{(2\pi)^3}(P + Q\cos\theta)\,\delta(\epsilon_o - \epsilon_1 - \epsilon_2)$$

$$= (2\pi)^{-5}\int p_1(\epsilon_1 + 1)p_2(\epsilon_2 + 1)\left[P(\epsilon_1,\epsilon_2) + Q(\epsilon_1,\epsilon_2)\cos\theta\right] \times$$

$$\times\, \delta(\epsilon_o - \epsilon_1 - \epsilon_2)\, d\Omega_1\, d\Omega_2\, d\epsilon_1\, d\epsilon_2 \tag{2.60}$$

In order to evaluate this integral, we first suppose that the momentum of one of the electrons, $\vec{p}_1$ say, is fixed, in the polar direction. This gives

$$d\Omega_2 = 2\pi \sin\theta \, d\theta$$

We then integrate over all possible directions of $\vec{p}_1$, which gives a factor of 4π. Since the emitted electrons are identical, we must insert an indistinguishability factor of $\frac{1}{2}$. Thus the partial rate for the emission of two electrons with momentum magnitudes p_1 and p_2 and an angle θ between their directions is given by

$$\frac{d\lambda^{(o)}}{d\epsilon_1 d\epsilon_2} = \frac{1}{2}(2\pi)^{-5}(4\pi)^2 \int_0^{\pi} (\epsilon_1 + 1)^2(\epsilon_2 + 1)^2\Big[P(\epsilon_1,\epsilon_2) + Q(\epsilon_1,\epsilon_2)\cos\theta\Big] \times$$

$$\times\, \delta(\epsilon_o - \epsilon_1 - \epsilon_2)\, \frac{1}{2}\sin\theta \, d\theta \tag{2.61}$$

$$= \frac{1}{2}(2\pi)^{-5}(4\pi)^2(\epsilon_1 + 1)^2(\epsilon_2 + 1)^2 P(\epsilon_1,\epsilon_2)\delta(\epsilon_o - \epsilon_1 - \epsilon_2) \tag{2.62}$$

using $p_i \approx E_i = \epsilon_i + 1$ $(i = 1, 2)$. From equations (2.50), (2.51) and (2.58) we get

$$\lambda^{(o)} = C \int_0^{\epsilon_o}\int_0^{\epsilon_o} (\epsilon_1 + 1)(\epsilon_2 + 1)\Big\{ f(\epsilon_1,\epsilon_2) + 4\, g(\epsilon_1,\epsilon_2) + 4\, h(\epsilon_1,\epsilon_2)\Big\} \times$$

$$\times\, \delta(\epsilon_o - \epsilon_1 - \epsilon_2)\, d\epsilon_1\, d\epsilon_2 \tag{2.63}$$

where
$$C = \frac{1}{2}\,\frac{(2\pi)^{-5}}{81}(G^2\eta)^2(3g_A^2 - g_V^2)^2\,|M_2|^2 \tag{2.64}$$

$$f(\epsilon_1,\epsilon_2) = (\epsilon_1 - \epsilon_2)^2\Big[(\epsilon_1+1)(\epsilon_2+1) - 1\Big]\Big[\frac{4}{R} - \frac{1}{\pi}(\epsilon_1+\epsilon_2+ 2\delta + 2\Big]^2 \tag{2.65}$$

$$g(\epsilon_1,\epsilon_2) = \Big[(\epsilon_1+ 1)(\epsilon_2+ 1) - 1\Big]\Big[\frac{1}{R} - \frac{1}{\pi}(\epsilon_1+ \epsilon_2+ 2\delta + 2)\Big]^2 \tag{2.66}$$

$$h(\epsilon_1,\epsilon_2) = (\epsilon_1 - \epsilon_2)^2\Big[\frac{4}{R} - \frac{1}{\pi}(\epsilon_1+\epsilon_2+ 2\delta + 2\Big]\Big[\frac{1}{R} - \frac{1}{\pi}(\epsilon_1+\epsilon_2+ 2\delta + 2\Big] \tag{2.67}$$

On evaluating (2.63) we obtain the following expression for the rate $\lambda^{(o)}$ of neutrinoless $\beta\beta$ decay :

$$\lambda^{(o)} = \frac{1}{2}\,\frac{(2\pi)^{-5}}{81}(G^2\eta)^2(3g_A^2 - g_V^2)^2 \;\times$$

$$\times\left\{\Big[\frac{4}{R} - \frac{1}{\pi}(\epsilon_o+ 2\delta + 2)\Big]^2 f(\epsilon_o) + 4\Big[\frac{1}{R} - \frac{1}{\pi}(\epsilon_o+ 2\delta + 2)\Big]^2 g(\epsilon_o) + \right.$$

$$\left. + 4\Big[\frac{4}{R} - \frac{1}{\pi}(\epsilon_o+ 2\delta + 2)\Big]\Big[\frac{1}{R} - \frac{1}{\pi}(\epsilon_o+ 2\delta + 2)\Big] h(\epsilon_o)\right\} |M_2|^2 \tag{2.68}$$

where

$$f(\epsilon_o) = \frac{1}{210}\,\epsilon_o^4\Big(\epsilon_o^3 + 14\epsilon_o^2 + 77\epsilon_o + 70\Big) \tag{2.69}$$

$$g(\epsilon_o) = \frac{1}{30}\,\epsilon_o^2\Big(\epsilon_o^3 + 10\epsilon_o^2 + 35\epsilon_o + 30\Big) \tag{2.70}$$

$$h(\epsilon_o) = \frac{1}{30}\,\epsilon_o^3\Big(\epsilon_o^2 + 10\epsilon_o + 10\Big) \tag{2.71}$$

So far we have neglected the effect of the daughter nucleus on the emitted electrons. To take account of this, we must give some consideration to the distortion of the electron wave function (from a plane wave) due to the Coulomb charge of the daughter nucleus. An appropriate correction for this effect may be made by insertion of the Fermi factor

$$F(Z,\ \epsilon) = \left[2\pi\alpha Z\,\frac{\epsilon+1}{p}\right]\left[1 - \exp\left(-\,2\pi\alpha Z\,\frac{\epsilon+1}{p}\right)\right]^{-1} \tag{2.72}$$

where $\alpha = \frac{1}{137}$ is the fine structure constant. Again using $p \approx \epsilon + 1$, we get

$$F(Z,\ \epsilon) = \frac{2\pi Z}{137}\left[1 - \exp\left(-\,\frac{2\pi Z}{137}\right)\right]^{-1} \tag{2.73}$$

where Z is the atomic number of the daughter nucleus. Also, inserting appropriate powers of m, c and $\hbar$ to ensure dimensional consistency, we obtain the final expression for the rate $\lambda^{(o)}$ of neutrinoless $\beta\beta$ decay :

$$\lambda^{(o)} = \frac{1}{2}\,\frac{(2\pi)^{-5}}{81}\,G^4\eta^2(3g_A^2 - g_V^2)^2\left(\frac{m^7c^4}{\hbar^{11}}\right)\left(\frac{2\pi Z}{137}\right)^2\left[1 - \exp\left(-\,\frac{2\pi Z}{137}\right)\right]^{-2} \times$$

$$\times\left\{\left[\frac{4}{R} - \frac{mc}{\pi\hbar}(\epsilon_o + 2\delta + 2)\right]^2 f(\epsilon_o) + 4\left[\frac{1}{R} - \frac{mc}{\pi\hbar}(\epsilon_o + 2\delta + 2)\right]^2 g(\epsilon_o)\right. +$$

$$\left. + 4\left[\frac{4}{R} - \frac{mc}{\pi\hbar}(\epsilon_o + 2\delta + 2)\right]\left[\frac{1}{R} - \frac{mc}{\pi\hbar}(\epsilon_o + 2\delta + 2)\right] h(\epsilon_o)\right\}|M_2|^2 \tag{2.74}$$

It may be of some interest to compare the expression (2.74) with the results quoted in the literature. Primakoff and Rosen(31) have obtained the following formula for the rate of neutrinoless decay :

$$\lambda^{(o)} = (2\pi)^{-5}\int_0^{\epsilon_o}\int_0^{\epsilon_o}(\epsilon_1 + 1)^2(\epsilon_2 + 1)^2 P'(\epsilon_1,\epsilon_2)\,\delta(\epsilon_o - \epsilon_1 - \epsilon_2)\,d\epsilon_1 d\epsilon_2 \tag{2.75}$$

where
$$P'(\epsilon_1,\epsilon_2) = \frac{16}{9R^2}\,g_A^4 G^4\eta^2(E_1 - E_2)^2\left(\frac{E_1E_2 - 1}{E_1E_2}\right)|M_2|^2$$

$$= \frac{16}{9R^2}\,g_A^4 G^4\eta^2(\epsilon_1 - \epsilon_2)^2\,\frac{(\epsilon_1+1)(\epsilon_2+1) - 1}{(\epsilon_1+1)(\epsilon_2+1)}\,|M_2|^2 \tag{2.76}$$

Thus according to these authors*

$$\lambda^{(o)} = (2\pi)^{-5} \frac{16}{9R^2} g_A^4 G^4 \eta^2 \; |M_2|^2 \times$$

$$\times \int_0^{\epsilon_o}\int_0^{\epsilon_o} (\epsilon_1 + 1)(\epsilon_2 + 1)(\epsilon_1 - \epsilon_2)^2 \left[(\epsilon_1 + 1)(\epsilon_2 + 1) - 1\right] (\epsilon_o - \epsilon_1 - \epsilon_2) d\epsilon_1 \, d\epsilon_2$$

$$= (2\pi)^{-5} \frac{16}{9R^2} g_A^4 G^4 \eta^2 \; f(\epsilon_o) \; |M_2|^2 \tag{2.77}$$

We see that of the three polynomials $f(\epsilon_o)$, $g(\epsilon_o)$ and $h(\epsilon_o)$ in equation (2.68) only $f(\epsilon_o)$ appears in (2.77). In Reference (31) the last term of equation (2.26), which contains the quantity $(E_1 + E_2 + 2)$, appears to have been omitted. As a result, equations (2.49) to (2.51) are modified by the substitutions

$$A = (E_1 - E_2)\left[\frac{4}{R} - \frac{E_1 + E_2 + 2\delta}{\pi}\right] \rightarrow A' = \frac{4}{R}(E_1 - E_2)$$

$$B = 2\left[\frac{1}{R} - \frac{E_1 + E_2 + 2\delta}{\pi}\right] \rightarrow B' = \frac{2}{R}$$

and this should give the following expression for the decay rate

$$\lambda^{(o)} = \frac{1}{2}(2\pi)^{-5} \frac{1}{81} G^4 \eta^2 (3g_A^2 - g_V^2)^2 \; |M_2|^2 \int_0^{\epsilon_o}\int_0^{\epsilon_o} (\epsilon_1 + 1)(\epsilon_2 + 1) \times \tag{2.78}$$

$$\times \left[(A'^2 + B'^2)\left(\frac{E_1 E_2 - 1}{E_1 E_2}\right) + 2\,A'B'\left(\frac{E_1 - E_2}{E_1 E_2}\right)\right] \delta(\epsilon_o - \epsilon_1 - \epsilon_2) \, d\epsilon_1 \, d\epsilon_2$$

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* For the purposes of this comparison we shall omit the Fermi correction factor.

$$\text{or } \lambda^{(o)} = \frac{1}{2}(2\pi)^{-5}\,\frac{1}{81}G^4\eta^2(3g_A^2 - g_V^2)^2\;|M_2|^2 \int_0^{\epsilon_o}\!\!\int_0^{\epsilon_o} (\epsilon_1 + 1)(\epsilon_2 + 1)\;\times$$

$$\times \left\{f'(\epsilon_1,\epsilon_2) + 4\,g'(\epsilon_1,\epsilon_2) + 4h'(\epsilon_1,\epsilon_2)\right\}\;\delta(\epsilon_o - \epsilon_1 - \epsilon_2)\; d\epsilon_1\, d\epsilon_2 \quad (2.79)$$

$$f'(\epsilon_1,\epsilon_2) = \left(\frac{4}{R}\right)^2(\epsilon_1 - \epsilon_2)^2\left[(\epsilon_1 + 1)(\epsilon_2 + 1) - 1\right]$$

$$g'(\epsilon_1,\epsilon_2) = \frac{1}{R^2}\left[(\epsilon_1 + 1)(\epsilon_2 + 1) - 1\right]$$

$$h'(\epsilon_1,\epsilon_2) = \frac{4}{R^2}(\epsilon_1 - \epsilon_2)^2.$$

Thus all three polynomials $f(\epsilon_o)$, $g(\epsilon_o)$ and $h(\epsilon_o)$ should still appear in the expression for $\lambda^{(o)}$. An examination of (2.77) shows that only $f(\epsilon_o)$ appears because, evidently, the following additional approximations have been made

$$B' \approx 0$$

$$3g_A^2 - g_V^2 \approx 3g_A^2$$

With these approximations, equation (2.78) becomes

$$\lambda^{(o)} = \frac{1}{2}(2\pi)^{-5}\,\frac{16}{81R^2}G^4\eta^2(9g_A^4)\;|M_2|^2 \int_0^{\epsilon_o}\!\!\int_0^{\epsilon_o} (\epsilon_1 + 1)(\epsilon_2 + 1)\;\times$$

$$\times\;(\epsilon_1 - \epsilon_2)^2\left[(\epsilon_1 + 1)(\epsilon_2 + 1) - 1\right]\delta(\epsilon_o - \epsilon_1 - \epsilon_2)\; d\epsilon_1\, d\epsilon_2$$

$$= \frac{1}{2}(2\pi)^{-5}\,\frac{16}{9R^2}g^4G^4\eta^2\; f(\epsilon_o)\;|M_2|^2 \quad (2.80)$$

This expression is smaller than the rate (2.77) given by Primakoff and Rosen by a factor of $\frac{1}{2}$.

An earlier calculation of the rate of neutrinoless $\beta\beta$ decay was made by Gruelling and Whitten[36]. By analogy with the nucleon current

$$J_\lambda = \overline{\Psi}_p \gamma_\lambda (1 + x\,\gamma_5)\Psi_n \; ; \; x = -\frac{g_A}{g_V}$$

the lepton current was assumed to be of the form

$$\begin{aligned} L_\lambda &= \overline{\Psi}_e \gamma_\lambda (1 + y\,\gamma_5)\Psi_\nu \\ &= \overline{\Psi}_e \gamma_\lambda \left[1 + (1 + \eta')\gamma_5\right] \Psi_\nu \; ; \; y = 1 + \eta' \end{aligned} \tag{2.81}$$

Neutrinoless $\beta\beta$ decay is then considered to arise from the creation of an incompletely polarized intermediate state Majorana neutrino which may be reabsorbed by the intermediate nucleus.

In order to examine the results of Reference (36) we shall write the lepton current (2.81) in the form

$$L_\lambda = \overline{\Psi}_e \gamma_\lambda \left[(1 + \tfrac{1}{2}\eta')(1 + \gamma_5) - \tfrac{1}{2}\eta'(1 - \gamma_5)\right] \Psi_\nu$$

A summation over the modes of the intermediate neutrino (see pp 34 - 35) then yields a factor

$$\begin{aligned} &-\eta'(\eta' + 2)\; \overline{u}(\vec{p}_1)\gamma_\lambda\left(1 + \frac{\vec{\alpha}\cdot\vec{k}}{k}\right)\gamma_4\gamma_\lambda\; u^*(\vec{p}_2) \\ &\approx -(2\eta')\; \overline{u}(\vec{p}_1)\gamma_\lambda\left(1 + \frac{\vec{\alpha}\cdot\vec{k}}{k}\right)\gamma_4\gamma_\lambda\; u^*(\vec{p}_2) \end{aligned}$$

(assuming $\eta' \ll 1$), as opposed to the factor

$$(2\eta)\ \bar{u}(\vec{p}_1)\gamma_\lambda\left(1 + \frac{\vec{\alpha}\cdot\vec{k}}{k}\right)\gamma_4\gamma_\lambda\ u^*(\vec{p}_2)$$

which is obtained by using the lepton current

$$L_\lambda = \bar{\Psi}_e\gamma_\lambda\left[(1+\gamma_5) + \eta(1-\gamma_5)\right]\Psi_\nu \tag{2.5}$$

We shall therefore assume that

$$-\eta' \approx \eta. \tag{2.82}$$

In Reference (36) the outgoing electrons are described by Coulomb wave functions and a finite neutrino mass m_ν is assumed. Using $m_\nu = 0$ and plane waves for the electrons the rate obtained by these authors becomes

$$\lambda^{(o)} = (2\pi)^{-5}\frac{G^4}{9}\eta'^2\left\{|X|^2 f_1(\epsilon_o) + |X'|^2 f_2(\epsilon_o) + 2\,\mathrm{Re}(XX'^*) f_3(\epsilon_o)\right\} \tag{2.83}$$

where

$$X = g_A^2(M_2 + M_3) + 2\,g_V^2\,M_1$$

$$X' = g_A^2(2M_2 - M_3) + g_V^2\,M_1$$

$$f_1(\epsilon_o) = \frac{1}{210}\,\epsilon_o^4\left(\epsilon_o^3 + 7\,\epsilon_o^2 + 75\,\epsilon_o + \frac{105}{2}\right) \neq f(\epsilon_o)$$

$$f_2(\epsilon_o) = \frac{1}{30}\,\epsilon_o^2(\epsilon_o^3 + 10\,\epsilon_o^2 + 35\,\epsilon_o + 30) = g(\epsilon_o)$$

$$f_3(\epsilon_o) = \frac{1}{30}\,\epsilon_o^3(\epsilon_o^2 + 10\,\epsilon_o + 10) = h(\epsilon_o)$$

We note that there is a slight discrepancy between the $f_1(\epsilon_o)$ of Reference (36) and the $f(\epsilon_o)$ of equation (2.69) and Reference (31). Making the assumptions $M_1 = -\frac{1}{3} M_2$ and $M_2 = M_3$ equation (2.83) reduces to

$$\lambda^{(o)} = (2\pi)^{-5} \frac{G^4}{81}\eta^2(3g_A^2 - g_V^2)^2 \left\{4\ f_1(\epsilon_o) + g(\epsilon_o) + 4\ h(\epsilon_o)\right\} |M_2|^2 \quad (2.84)$$

This is to be compared with equation (2.68)

$$\lambda^{(o)} = \frac{1}{2}(2\pi)^{-5} \frac{G^4}{81}\eta^2(3g_A^2 - g_V^2)\ |M_2|^2 \times$$

$$\left\{ \left[\frac{4}{R} - \frac{1}{\pi}(\epsilon_o + 2\delta + 2)\right]^2 f(\epsilon_o) + 4\left[\frac{1}{R} - \frac{1}{\pi}(\epsilon_o + 2\delta + 2)\right]^2 g(\epsilon_o) + \right.$$

$$\left. + 4\left[\frac{4}{R} - \frac{1}{\pi}(\epsilon_o + 2\delta + 2)\right]\left[\frac{1}{R} - \frac{1}{\pi}(\epsilon_o + 2\delta + 2)\right] h(\epsilon_o)\right\} \quad (2.68)$$

In the approximation

$$\frac{1}{R} \gg \frac{1}{\pi}(\epsilon_o + 2\delta + 2) \quad (2.85)$$

equation (2.68) reduces to

$$\lambda^{(o)} = 2(2\pi)^{-5} \frac{G^4\eta^2}{81R^2}(3g_A^2 - g_V^2)^2 \left\{4\ f(\epsilon_o) + g(\epsilon_o) + 4\ h(\epsilon_o)\right\} |M_2|^2 \quad (2.86)$$

Apart from the discrepancy between $f_1(\epsilon_o)$ and $f(\epsilon_o)$, the expression (2.86) is larger by a factor of 2 than the result (2.84) obtained from Reference (36).

We note that in References (31) and (36) the energy gap δ has been omitted from the calculation. In (31) the quantity $E_i + \delta + k$ $(i = 1, 2)$

has been replaced by k, while in (36) the approximation (2.85) has apparently been used. When the numerical constants m, c and $\hbar$ are inserted, this inequality becomes

$$\frac{1}{R} \gg 2\left(\frac{mc}{\hbar}\right)(\epsilon_o + 2\delta + 2)$$

where $$R = 1.2\, A^{1/3} \times 10^{-13}\,\text{cm}.$$

However, there seems to be no a priori reason to make these approximations, especially in light of the estimate $\delta \leq 20$ made by these authors. For example, using $A^{1/3} \approx 5$, $\epsilon_o = 5$ and $\delta = 10$, we get

$$\frac{1}{R} \approx 2 \times 10^{12}\ \text{cm}^{-1}$$

$$2\left(\frac{mc}{\hbar}\right)(\epsilon_o + 2\delta + 2) \approx 0.3 \times 10^{12}\ \text{cm}^{-1}$$

The energy gap δ is, in general, not known but as we shall see later, the magnitude of the parameter η is rather sensitive to the value of δ.

CHAPTER III

THE TWO-NEUTRINO PROCESS

In this Chapter we shall calculate the rate $\lambda^{(2)}$ of the two-neutrino process. This differs from the no-neutrino process in the important respect that there are now four leptons (e_1, e_2, $\overline{\nu}_1$, $\overline{\nu}_2$) in the final state, and we can return to the usual two-component form of the leptonic weak current

$$L_\lambda = \overline{\Psi}_e \gamma_\lambda (1 + \gamma_5) \Psi_\nu \qquad (1.33)$$

The mechanism responsible for the two-neutrino process can be represented by the diagrams shown in Figure 2. The matrix element corresponding to the first process is

$$M^{(i)} = \frac{G^2}{2} \int\int d^3\vec{x}\, d^3\vec{y} \sum_N$$

$$\left\{ \frac{\langle e_2\nu_2|L_\mu(\vec{y})|0\rangle\langle N_f|J_\mu(\vec{y})|N\rangle\langle e_1\nu_1|L_\lambda(\vec{x})|0\rangle\langle N|J_\lambda(\vec{x})|N_i\rangle}{k_1 + E_1 + W_N - W_i} + \right.$$

$$\left. + \frac{\langle e_1\nu_1|L_\lambda(\vec{x})|0\rangle\langle N_f|J_\lambda(\vec{x})|N\rangle\langle e_2\nu_2|L_\mu(\vec{y})|0\rangle\langle N|J_\mu(\vec{y})|N_i\rangle}{k_2 + E_2 + W_N - W_i} \right\} \qquad (3.1)$$

where k_j, E_j ($j = 1, 2$) are the energies of the j^{th} neutrino and the j^{th} electron, respectively ; W_N is the energy of the intermediate nucleus,

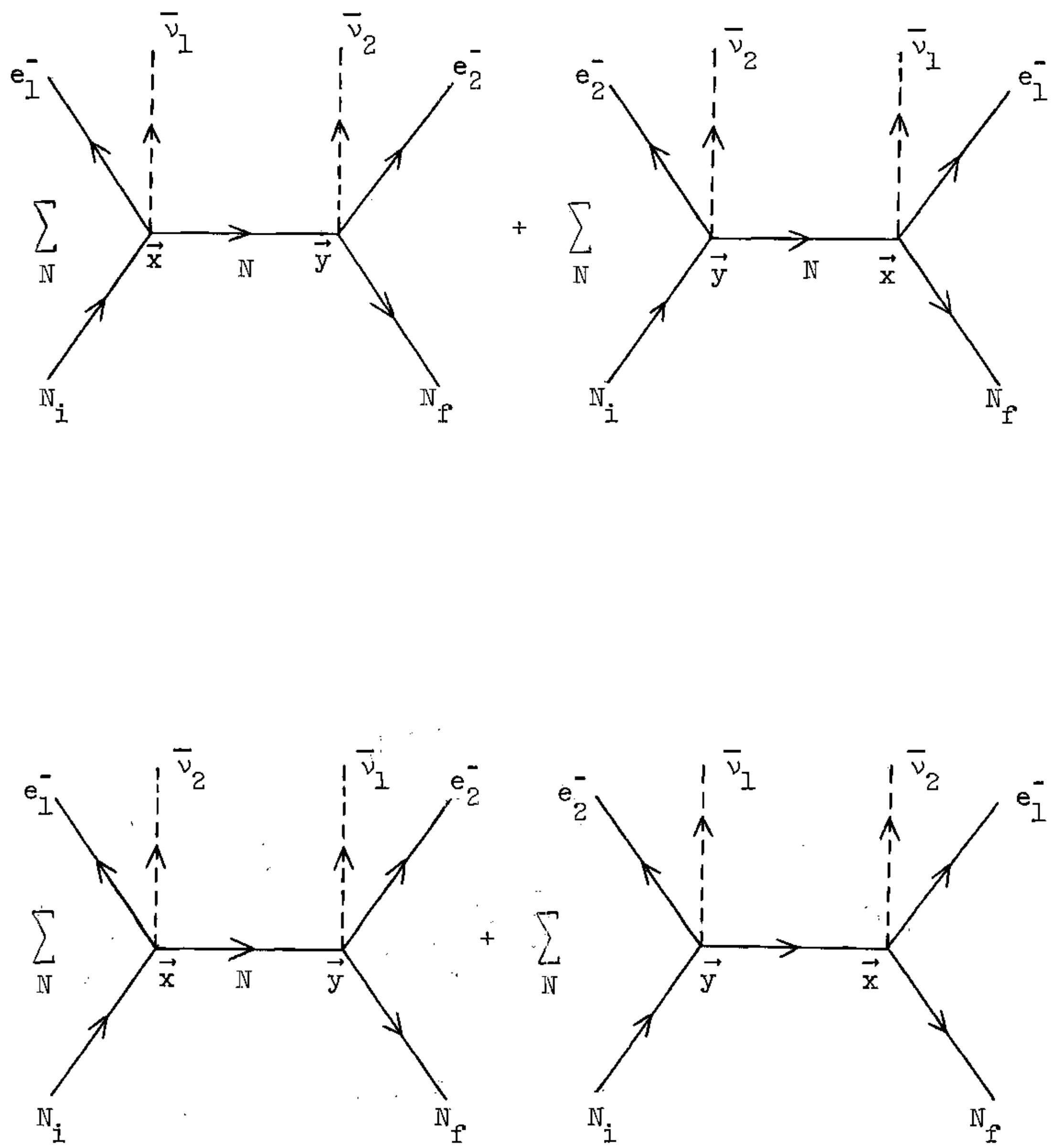

Figure 2. Second Order Weak Diagrams for Two-neutrino ββ Decay

and W_i is the energy of the initial nucleus. We shall again replace W_N by an average value $\langle W_N \rangle$. Then performing the closure summation over the complete set of intermediate nuclear states, we get

$$M^{(i)} = \frac{G^2}{2} \int\int d^3\vec{x}\, d^3\vec{y}\; e^{-i(\vec{p}_1\cdot\vec{x} + \vec{p}_2\cdot\vec{y})}\; e^{+i(\vec{k}_1\cdot\vec{x} + \vec{k}_2\cdot\vec{y})} \times$$

$$\left\{ \frac{\left[\bar{u}(\vec{p}_2)\gamma_\mu(1+\gamma_5)u(\vec{k}_2)\right]\left[\bar{u}(\vec{p}_1)\gamma_\lambda(1+\gamma_5)u(\vec{k}_1)\right]}{k_1 + E_1 + \delta} + \right.$$

$$\left. + \frac{\left[\bar{u}(\vec{p}_1)\gamma_\lambda(1+\gamma_5)u(\vec{k}_1)\right]\left[\bar{u}(\vec{p}_2)\gamma_\mu(1+\gamma_5)u(\vec{k}_2)\right]}{k_2 + E_2 + \delta} \right\} \times$$

$$\times \quad \langle N_f | J_\lambda(\vec{x})\; J_\mu(\vec{y}) | N_i \rangle \tag{3.2}$$

where $\langle W_N \rangle - W_i \equiv \delta$, and we have used the fact that $J_\lambda(\vec{x})$ and $J_\mu(\vec{y})$ commute. As before, we take

$$J_\lambda(\vec{x}) = \sum_n \tau_n^{(+)} (\Gamma_\lambda)_n\; \delta^3(\vec{x} - \vec{r}_n) \tag{2.21}$$

$$J_\mu(\vec{y}) = \sum_m \tau_m^{(+)} (\Gamma_\mu)_m\; \delta^3(\vec{y} - \vec{r}_m) \tag{2.25}$$

where $$(\Gamma_\lambda)_n = g_V\, \delta_{\lambda 4} + i\, g_A (\sigma_n)_\lambda (1 - \delta_{\lambda 4}). \tag{2.24}$$

On carrying out the $\vec{x}$ and $\vec{y}$ integration, we get

$$M^{(i)} = \frac{G^2}{2} K \left[\overline{u}(\vec{p}_1)\gamma_\lambda(1 + \gamma_5)u(\vec{k}_1)\right]\left[\overline{u}(\vec{p}_2)\gamma_\mu(1 + \gamma_5)u(\vec{k}_2)\right] \times$$

$$\times \langle N_f | \sum_{n,m} \tau_n^{(+)} \tau_m^{(+)} (\Gamma_\lambda)_n (\Gamma_\mu)_m \, e^{-i(\vec{p}_1 \cdot \vec{r}_n + \vec{p}_2 \cdot \vec{r}_m)} e^{i(\vec{k}_1 \cdot \vec{r}_n + \vec{k}_2 \cdot \vec{r}_m)} | N_i \rangle \tag{3.3}$$

where
$$K \equiv \frac{1}{k_1 + E_1 + \delta} + \frac{1}{k_2 + E_2 + \delta} \tag{3.4}$$

Since we are considering only $0^+ \to 0^+$ transitions, the operator sandwiched between the nuclear states must be invariant under rotations and inversions of the nuclear coordinates. We therefore replace the exponential terms by unity, to give

$$M^{(i)} = \frac{G^2}{2} K \, (A_\lambda B_\mu) \, \langle N_f | \sum_{n,m} \tau_n^{(+)} \tau_m^{(+)} (\Gamma_\lambda)_n (\Gamma_\mu)_m | N_i \rangle \tag{3.5}$$

where
$$A_\lambda = \overline{u}(\vec{p}_1)\gamma_\lambda(1 + \gamma_5)u(\vec{k}_1) \tag{3.6}$$

$$B_\mu = \overline{u}(\vec{p}_2)\gamma_\mu(1 + \gamma_5)u(\vec{k}_2) \tag{3.7}$$

and
$$(\Gamma_\lambda)_n(\Gamma_\mu)_m = g_V^2 \, \delta_{\lambda 4} \, \delta_{\mu 4} - g_A^2 (\sigma_\lambda)_n (\sigma_\mu)_m (1 - \delta_{\lambda 4})(1 - \delta_{\mu 4}) +$$

$$+ i \, g_V g_A \left[\delta_{\lambda 4} (\sigma_\mu)_m (1 - \delta_{\mu 4}) + \delta_{\mu 4} (\sigma_\lambda)_n (1 - \delta_{\lambda 4}) \right] . \tag{2.27}$$

We shall write equation (3.5) symbolically as

$$M^{(i)} = \frac{G^2}{2} K \, A_\lambda B_\mu \, (\Gamma_\lambda)_n (\Gamma_\mu)_m \tag{3.8}$$

In other words, in (3.8) $(\Gamma_\lambda)_n(\Gamma_\mu)_m$ is a shorthand notation for

$$\langle N_f | \sum_{n,m} \tau_n^{(+)} \tau_m^{(+)} (\Gamma_\lambda)_n (\Gamma_\mu)_m | N_i \rangle$$

Thus we may write

$$M^{(i)} = \frac{G^2}{4} K\, A_\lambda B_\mu \left[(\Gamma_\lambda)_n (\Gamma_\mu)_m + (\Gamma_\mu)_n (\Gamma_\lambda)_m \right] \tag{3.9}$$

Since the expression in square brackets is symmetric in the indices λ and μ, $A_\lambda B_\mu$ may be replaced by its symmetric part, to give

$$M^{(i)} = \frac{G^2}{8} K \left(A_\lambda B_\mu + A_\mu B_\lambda \right) \left[(\Gamma_\lambda)_n (\Gamma_\mu)_m + (\Gamma_\mu)_n (\Gamma_\lambda)_m \right] \tag{3.10}$$

As explained in Appendix II, we need only consider the cases $\lambda = 4$; $\mu = 4$ and $\lambda = j$; $\mu = k$. The first case gives

$$M^{(i)}(\lambda = 4 \ ; \ \mu = 4) = \frac{1}{8} G^2 K \left(2A_4 B_4 \right) \left[2(\Gamma_4)_n (\Gamma_4)_m \right]$$

$$= \frac{1}{2} g_V^2 \, G^2 K \left(A_4 B_4 \right) M_1 \tag{3.11}$$

where

$$M_1 \equiv \langle N_f | \sum_{n,m} \tau_n^{(+)} \tau_m^{(+)} | N_i \rangle \tag{2.32}$$

$$M^{(i)}(\lambda = j \ ; \ \mu = k) = -\frac{1}{8} g_A^2 \, G^2 K \left(A_j B_k + A_k B_j \right) \left[(\sigma_j)_n (\sigma_k)_m + (\sigma_k)_n (\sigma_j)_m \right] \tag{3.12}$$

$$= -\frac{1}{8} g_A^2 \, G^2 K \left(R_{jk} + \frac{2}{3} \delta_{jk} \, \vec{A} \cdot \vec{B} \right) \left[S_{jk} + \frac{2}{3} \delta_{jk} \, (\vec{\sigma}_n \cdot \vec{\sigma}_m) \right] \tag{3.13}$$

where R_{jk} and S_{jk} are irreducible tensors defined by

$$R_{jk} \equiv A_j B_k + A_k B_j - \frac{2}{3} \delta_{jk} \, \vec{A} \cdot \vec{B}$$

$$S_{jk} \equiv (\sigma_j)_n(\sigma_k)_m + (\sigma_k)_n(\sigma_j)_m - \frac{2}{3}\,\delta_{jk}\,\vec{\sigma}_n\cdot\vec{\sigma}_m \tag{3.15}$$

The tensors R_{jk} and S_{jk} are traceless ($R_{jj} = S_{jj} = 0$) and so the only nucleon scalar available from (3.13) is

$$-\frac{1}{8}\,g_A^2\,G^2K\left(\frac{2}{3}\,\delta_{jk}\,\vec{A}\cdot\vec{B}\right)\left(\frac{2}{3}\,\delta_{jk}\,\vec{\sigma}_n\cdot\vec{\sigma}_m\right) = -\frac{1}{6}\,g_A^2\,G^2K\,(\vec{A}\cdot\vec{B})\,M_2$$

where
$$M_2 = \langle N_f|\sum_{n,m}\tau_n^{(+)}\tau_m^{(+)}\,\vec{\sigma}_n\cdot\vec{\sigma}_m|N_i\rangle \tag{2.33}$$

Thus for $0^+ \to 0^+$ two-neutrino transitions we have

$$M^{(i)} = \frac{1}{2}\,G^2K\left\{g_V^2\left[\bar{u}(\vec{p}_1)\gamma_4(1+\gamma_5)u(\vec{k}_1)\right]\left[\bar{u}(\vec{p}_2)\gamma_4(1+\gamma_5)u(\vec{k}_2)\right]M_1 - \right.$$

$$\left. -\frac{1}{3}\,g_A^2\left[\bar{u}(\vec{p}_1)\vec{\gamma}(1+\gamma_5)u(\vec{k}_1)\right]\cdot\left[\bar{u}(\vec{p}_2)\vec{\gamma}(1+\gamma_5)u(\vec{k}_2)\right]M_2\right\} \tag{3.16}$$

The matrix element $M^{(ii)}$ corresponding to the second diagram in Figure 2 is obtained from $M^{(i)}$ by interchanging the roles of $\bar{\nu}_1$ and $\bar{\nu}_2$, i.e.

$$M^{(ii)} = P_{12}\,M^{(i)}$$

$$M^{(ii)} = -\frac{1}{2}\,G^2L\left\{g_V^2\left[\bar{u}(\vec{p}_2)\gamma_4(1+\gamma_5)u(\vec{k}_1)\right]\left[\bar{u}(\vec{p}_1)\gamma_4(1+\gamma_5)u(\vec{k}_2)\right]M_1 - \right.$$

$$\left. -\frac{1}{3}\,g_A^2\left[\bar{u}(\vec{p}_2)\vec{\gamma}(1+\gamma_5)u(\vec{k}_1)\right]\cdot\left[\bar{u}(\vec{p}_1)\vec{\gamma}(1+\gamma_5)u(\vec{k}_2)\right]M_2\right\} \tag{3.17}$$

where
$$L = \frac{1}{k_2 + E_1 + \delta} + \frac{1}{k_1 + E_2 + \delta} \tag{3.18}$$

The total matrix element for two-neutrino $\beta\beta$ decay is then given by

$$M = (1 - P_{12})\, M^{(i)}$$

$$= \frac{1}{2} G^2 K \left\{g_V^2\, P\, M_1 - \frac{1}{3} g_A^2\, Q\, M_2\right\} - \frac{1}{2} G^2 L \left\{g_V^2\, P' M_1 - \frac{1}{3} g_A^2\, Q' M_2\right\} \quad (3.19)$$

where $$P = \left[\bar{u}(\vec{p}_1)\gamma_4(1 + \gamma_5)u(\vec{k}_1)\right]\left[\bar{u}(\vec{p}_2)\gamma_4(1 + \gamma_5)u(\vec{k}_2)\right] \quad (3.20a)$$

$$Q = \left[\bar{u}(\vec{p}_1)\vec{\gamma}(1 + \gamma_5)u(\vec{k}_1)\right]\cdot\left[\bar{u}(\vec{p}_2)\vec{\gamma}(1 + \gamma_5)u(\vec{k}_2)\right] \quad (3.21a)$$

P' and Q' are obtained from P and Q, respectively, by interchanging $\vec{p}_1$ and $\vec{p}_2$:

$$P' = \left[\bar{u}(\vec{p}_2)\gamma_4(1 + \gamma_5)u(\vec{k}_1)\right]\left[\bar{u}(\vec{p}_1)\gamma_4(1 + \gamma_5)u(\vec{k}_2)\right] \quad (3.20b)$$

$$Q' = \left[\bar{u}(\vec{p}_2)\vec{\gamma}(1 + \gamma_5)u(\vec{k}_1)\right]\cdot\left[\bar{u}(\vec{p}_1)\gamma(1 + \gamma_5)u(\vec{k}_2)\right] \quad (3.21b)$$

Equation (3.19) may be written as

$$M = \frac{1}{2} g_V^2\, G^2(K\, P - L\, P')\, M_1 - \frac{1}{6} g_A^2\, G^2(K\, Q - L\, Q')\, M_2 \quad (3.22)$$

We must now evaluate the quantity $\sum |M|^2$, where $\sum$ denotes summation over the spin orientations of the emitted electrons and neutrinos. We will then take the average over the directions of the neutrino momenta and denote the result by $\sum_{\text{ave}\, \vec{k}_1, \vec{k}_2}$. This gives (see Appendix III)

$$\sum_{\text{ave } \vec{k}_1, \vec{k}_2} | M |^2 = g_V^4 G^4 \left[(K^2 + L^2) - KL \left(1 + \frac{\vec{p}_1 \cdot \vec{p}_2}{E_1 E_2}\right)\right] | M_1 |^2 +$$

$$+ \frac{1}{3} g_A^4 G^4 \left[(K^2 + L^2)\left(1 - \frac{2}{3} \frac{\vec{p}_1 \cdot \vec{p}_2}{E_1 E_2}\right) + KL \left(1 - \frac{5}{3} \frac{\vec{p}_1 \cdot \vec{p}_2}{E_1 E_2}\right)\right] | M_2 |^2 -$$

$$- 2 g_V^2 g_A^2 G^4 KL \left(1 - \frac{1}{3} \frac{\vec{p}_1 \cdot \vec{p}_2}{E_1 E_2}\right) \mathrm{Re}(M_1^* M_2) \qquad (3.23)$$

where
$$K = \frac{1}{k_1 + E_1 + \delta} + \frac{1}{k_2 + E_2 + \delta} \qquad (3.4)$$

$$L = \frac{1}{k_2 + E_1 + \delta} + \frac{1}{k_1 + E_2 + \delta} \qquad (3.18)$$

We shall make the further approximation of neglecting the distinctions among the lepton energies in the energy denominators of (3.4) and (3.18). The energy spectrum of $k_i + E_j (i,j = 1,2)$ must be symmetric in the indices i, j and hence the spectrum average of $k_i + E_j$ must be independent of the indices i and j, and we take this average to be $\frac{1}{2} E_o$,

where $E_o = k_1 + k_2 + E_1 + E_2 = k_1 + k_2 + (\epsilon_1 + 1) + (\epsilon_2 + 1)$ (3.24)

ϵ_1 and ϵ_2 being the kinetic energies of the electrons. The energy release for the decay, i.e. the maximum kinetic energy of either electron, is then

$$\epsilon_o = E_o - 2 \qquad (3.25)$$

Thus we write

$$K \approx L \approx \frac{2}{\left[\frac{1}{2}(\epsilon_o + 2) + \delta\right]} = \frac{4}{\epsilon_o + 2\delta + 2} \tag{3.26}$$

With this approximation (3.23) becomes

$$\sum_{\text{ave } \vec{k}_1, \vec{k}_2} |M|^2 = \frac{1}{9} G^4 K^2 \left[(g_V^4 + 9g_A^4)\left(1 - \frac{\vec{p}_1 \cdot \vec{p}_2}{E_1 E_2}\right) + 6g_V^2 g_A^4 \left(1 - \frac{1}{3}\frac{\vec{p}_1 \cdot \vec{p}_2}{E_1 E_2}\right)\right] |M_2|^2 \tag{3.27}$$

where we have used

$$M_1 = -\frac{1}{3} M_2 \tag{2.42}$$

We shall write equation (3.27) as

$$\sum_{\text{ave } \vec{k}_1, \vec{k}_2} |M|^2 = \frac{1}{9} G^4 K^2 \left(A - B \frac{\vec{p}_1 \cdot \vec{p}_2}{E_1 E_2}\right) \tag{3.28}$$

where

$$A = (g_V^2 + 3g_A^2)^2 \, |M_2|^2 \tag{3.29}$$

$$B = (g_V^4 + 2g_V^2 g_A^2 + 9g_A^4) \, |M_2|^2 \tag{3.30}$$

The partial rate for the production of two electrons with their momentum magnitudes in the range $dp_1 dp_2$ and an angle θ between their emission directions is given by

$$d\lambda^{(2)} = 2\pi \frac{d^3\vec{p}_1}{(2\pi)^3}\frac{d^3\vec{p}_2}{(2\pi)^3}\frac{d^3\vec{k}_1}{(2\pi)^3}\frac{d^3\vec{k}_2}{(2\pi)^3} \delta(\epsilon_o - \epsilon_1 - \epsilon_2 - k_1 - k_2) \sum_{\text{ave } \vec{k}_1, \vec{k}_2} |M|^2$$

$$= (2\pi)^{-11} p_1^2 dp_1 p_2^2 dp_2 k_1^2 dk_1 k_2^2 dk_2 d\Omega^{(e_1)} d\Omega^{(e_2)} d\Omega^{(k_1)} d\Omega^{(k_2)} \sum_{\text{ave } \vec{k}_1, \vec{k}_2} |M|^2 \times$$

$$\times \; \delta(\epsilon_o - \epsilon_1 - \epsilon_2 - k_1 - k_2) \tag{3.31}$$

Integrating over $d\Omega^{(k_1)}$ and $d\Omega^{(k_2)}$ gives a factor of $(4\pi)^2$. The integration over the electron directions is carried out by considering $\vec{p}_2$, say, to be fixed and integrating over the directions of $\vec{p}_1$ relative to $\vec{p}_2$ (this gives a factor of $2\pi \sin\theta \, d\theta$) ; we then integrate over all possible directions of $\vec{p}_2$, which gives an additional factor of 4π. Since two pairs of identical particles are emitted, an indistinguishability factor of $\frac{1}{4}$ must also be included. Thus we obtain from (3.31)

$$\frac{d\lambda^{(2)}}{dp_1 dp_2} = \frac{4}{9} (2\pi)^{-7} G^4 K^2 \int p_1^2 p_2^2 k_1^2 k_2^2 \, \delta(\epsilon_o - \epsilon_1 - \epsilon_2 - k_1 - k_2) \, dk_1 \, dk_2 \; \times$$

$$\times \int_0^\pi (A - B \cos\theta) \frac{1}{2} \sin\theta \, d\theta \tag{3.32}$$

$$= \frac{4}{9}(2\pi)^{-7} G^4 K^2 \, A \int_0^{\epsilon_o} \int_0^{\epsilon_o} (\epsilon_1 + 1)^2 (\epsilon_2 + 1)^2 k_1^2 k_2^2 \, \delta(\epsilon_o - \epsilon_1 - \epsilon_2 - k_1 - k_2) dk_1 dk_2 \tag{3.33}$$

Using the δ-function to perform the k_2 integration, we get

$$\frac{d\lambda^{(2)}}{d\epsilon_1 d\epsilon_2} = \frac{4}{9}(2\pi)^{-7} G^4 K^2 \, A(\epsilon_1 + 1)^2 (\epsilon_2 + 1)^2 \int_0^{\epsilon_o - \epsilon_1 - \epsilon_2} k_1^2 (\epsilon_o - \epsilon_1 - \epsilon_2 - k_1)^2 dk_1 \tag{3.34}$$

$$= \frac{4}{9}(2\pi)^{-7} G^4 K^2 \, A \, (\epsilon_1 + 1)^2 (\epsilon_2 + 1)^2 \frac{(\epsilon_o - \epsilon_1 - \epsilon_2)^5}{30} \tag{3.35}$$

The partial rate for the emission of one electron, e_1 say, with kinetic energy in the range $d\epsilon_1$, is given by

$$\frac{d\lambda^{(2)}}{d\epsilon_1} = \frac{4}{9}(2\pi)^{-7}G^4K^2\, A\, \frac{(\epsilon_1+1)^2}{30} \int_0^{\epsilon_0-\epsilon_1} (\epsilon_2+1)^2(\epsilon_0-\epsilon_1-\epsilon_2)^5 d\epsilon_2 \qquad (3.36)$$

whence

$$d\lambda^{(2)} = \frac{4}{9}(2\pi)^{-7}G^4K^2A(\epsilon_1+1)^2(\epsilon_0-\epsilon_1)^6 \frac{1}{7!}\left[(\epsilon_0-\epsilon_1)^2+8(\epsilon_0-\epsilon_1)+28\right] d\epsilon_1 \qquad (3.37)$$

The total decay rate is obtained by integration of (3.37). Inserting the Coulomb correction (2.73), we obtain

$$\lambda^{(2)} = \frac{16}{9\times 7!}(2\pi)^{-7}G^4K^2\, A\left(\frac{2\pi Z}{137}\right)^2\left[1-\exp\left(-\frac{2\pi Z}{137}\right)\right]^{-2} k(\epsilon_0) \qquad (3.38)$$

where $$k(\epsilon_0) = \epsilon_0^7\left(1+\frac{1}{2}\epsilon_0+\frac{1}{9}\epsilon_0^2+\frac{1}{90}\epsilon_0^3+\frac{1}{1980}\epsilon_0^4\right) \qquad (3.39)$$

Finally, substituting for K and A from (3.26) and (3.29) and inserting appropriate powers of m, $\hbar$ and c to restore dimensional consistency, we get

$$\lambda^{(2)} = \frac{16}{9!}\frac{G^4}{\pi^7\hbar^2c}\left(\frac{mc}{\hbar}\right)^{11}\frac{1}{m^2c^4}\left(3g_A^2+g_V^2\right)^2\left(\frac{2\pi Z}{137}\right)^2\left[1-\exp\left(-\frac{2\pi Z}{137}\right)\right]^{-2} \times$$

$$\times \frac{k(\epsilon_0)}{(\epsilon_0+2\delta+2)^2}\,|M_2|^2 \qquad (3.40)$$

This expression agrees with the result obtained by Konopinski[37]. However, we are unable to obtain the result quoted by Gruelling and

Whitten[36], which is larger than (3.40) by a factor of four.

CHAPTER IV

COMPETITION BETWEEN NEUTRINOLESS AND TWO-NEUTRINO DOUBLE BETA DECAY

Combining equations (2.74) and (3.40), we obtain the following expression for the total rate $\lambda = \lambda^{(o)} + \lambda^{(2)}$ of double beta decay :

$$\lambda = \frac{32}{81}\pi^6 G^4\left(\frac{m^7c^4}{h^{11}}\right)\left(\frac{2\pi Z}{137}\right)^2\left[1 - \exp\left(-\frac{2\pi Z}{137}\right)\right]^{-2} \times$$

$$\times\left\{\eta^2(3g_A^2 - g_V^2)^2\left[P^2 f(\epsilon_o) + 4\,Q^2 g(\epsilon_o) + 4\,PQ\,h(\epsilon_o)\right] + \right.$$

$$\left. + \frac{32}{35}(3g_A^2 + g_V^2)^2\left(\frac{mc}{h}\right)^2\frac{k(\epsilon_o)}{(\epsilon_o + 2\delta + 2)^2}\right\}|M_2|^2 \tag{4.1}$$

where

$$P = \frac{4}{R} - 2\left(\frac{mc}{h}\right)\left(\epsilon_o + 2\delta + 2\right) \tag{4.2}$$

$$Q = \frac{1}{R} - 2\left(\frac{mc}{h}\right)\left(\epsilon_o + 2\delta + 2\right) \tag{4.3}$$

$$R = \left(1.2\,A^{1/3}\right)10^{-13}\ \text{cm} \tag{4.4}$$

$$f(\epsilon_o) = \frac{1}{210}\epsilon_o^4\left(\epsilon_o^3 + 14\,\epsilon_o^2 + 77\,\epsilon_o + 70\right) \tag{2.69}$$

$$g(\epsilon_o) = \frac{1}{30}\epsilon_o^2\left(\epsilon_o^3 + 10\,\epsilon_o^2 + 35\,\epsilon_o + 30\right) \tag{2.70}$$

$$h(\epsilon_o) = \frac{1}{30}\epsilon_o^3\left(\epsilon_o^2 + 10\,\epsilon_o + 10\right) \tag{2.71}$$

$$k(\epsilon_o) = \epsilon_o^7\left(1 + \frac{1}{2}\epsilon_o + \frac{1}{9}\epsilon_o^2 + \frac{1}{90}\epsilon_o^3 + \frac{1}{1980}\epsilon_o^4\right) \tag{3.39}$$

The half-life of the decay is given by

$$\tau = \frac{\ln 2}{\lambda} \text{ sec} = \frac{\ln 2}{\lambda} \frac{1}{3.156 \times 10^{7}} \text{ years}$$

$$\equiv 10^{T} \text{ years}$$

or

$$\lambda = \frac{\ln 2}{3.156 \times 10^{7} \times 10^{T}} \text{ year}^{-1} \tag{4.5}$$

and substituting this into (4.1), we get

$$\eta^2 = \frac{1}{(3g_A^2 - g_V^2)^2 \left[P^2 f(\epsilon_o) + 4\, Q^2 g(\epsilon_o) + 4\, PQ\, h(\epsilon_o)\right]} \times$$

$$\times \left\{ \frac{81}{32} \left(\frac{137}{2\pi Z}\right)^2 h^{11} \frac{\left[1 - \exp\left(-\frac{2\pi Z}{137}\right)\right]}{G^4 \pi^6 m^7 c^4} \times \frac{\ln 2}{3.156 \times 10^7 \times 10^T} \times \frac{1}{|M_2|^2} \right. -$$

$$\left. - \frac{32}{35} \left(3g_A^2 + g_V^2\right)^2 \left(\frac{mc}{h}\right)^2 \frac{k(\epsilon_o)}{(\epsilon_o + 2\delta + 2)^2} \right\} \tag{4.6}$$

We give here the values of the numerical constants which will be needed in subsequent calculations :

$$h = 6.626 \times 10^{-27} \text{ erg-sec}$$

$$m = 9.1 \times 10^{-28} \text{ gm}$$

$$c = 3.0 \times 10^{10} \text{ cm/sec}$$

$$G = 1.435 \times 10^{-49} \text{ erg cm}^3$$

$$g_V = 1 \; ; \; g_A = 1.2$$

The greatest uncertainties in evaluating the parameter η stem from our ignorance of the energy levels of those intermediate nuclear states with spin 0 or 1, the magnitudes of the nuclear matrix elements and, to a lesser extent, the experimental lifetimes of the decays. It will be seen from the decay schemes of the nuclei under consideration that, with the exception of ^{128}Te, little information on the values of the energy gap δ is available. In the case of ^{130}Te, which has a half-life of $10^{21.34}$ years, a lower limit can be set on the value of δ by imposing the requirement that η be real*. Alternatively, we may compare the theoretical values of the two-neutrino rate $\lambda^{(2)}$ (as a function of δ) with the known overall experimental rate $\lambda = 3.2 \times 10^{-22}$ year^{-1}, and use the fact that $\lambda^{(2)}$ must be less than λ. There is as yet no firm experimental data on the half-lives of ^{48}Ca and ^{128}Te. In the former case only lower limits are available, the most recent being those of Bardin et al.[18] who found the half-life of the no-neutrino process to be $> 2 \times 10^{21}$ years and the half-life of the two-neutrino process to be $> 3.6 \times 10^{19}$ years. In the latter case the half-life of $10^{22.5}$ years obtained by Takaoka and Ogata[16] is subject to a possibly large experimental error.

We must now examine the important question of the magnitudes of the nuclear matrix elements, about which there is still considerable theoretical uncertainty. The usual procedure has been to separate out

* It has been shown by Primakoff and Sharp[33] that a Majorana neutrino coupled within the lepton weak current in such a way that it appears in a linear combination of helicity states with relative weight η (η complex) describes both lepton number and CP violation. Since we are not concerned here with the question of CP violation, we shall suppose that η is real.

the dimensional factor R^{-1} (R is the nuclear radius) and to assume that the remaining dimensionless integral over the nuclear variables is constant for all nuclei and has a value ≈ 0.1. As Khodel[34] has pointed out, however, this ignores the possibility of fluctuations in the magnitudes of the nuclear matrix elements due to variations in the number of nucleons in the nucleus. He has suggested a method for the evaluation of the nuclear matrix elements of $\beta\beta$ decay, using the theory of finite Fermi systems[35]. But even in the simplest case, viz. a pair of neutrons outside closed shells in the orbit $\nu_1 (\nu = n, \ell, j)$ with total angular momentum I decaying into a pair of protons in the orbit ν_2 with the same angular momentum, the calculations are quite involved, and an evaluation in the case of the more complex nuclei under consideration here will not be attempted. The estimates made by Khodel for the decay of ^{48}Ca (which falls into the "simplest" category) are as follows :

$$\begin{aligned} M_1 &= -0.05 \\ M_2 &= +0.18 \\ M_3 &\approx 0 \end{aligned} \tag{4.7}$$

These estimates tend to confirm the approximation

$$M_1 = -\frac{1}{3} M_2 \tag{2.42}$$

made earlier, but show appreciable variations from the conventional estimate of 0.1. There is no simple way to evaluate the matrix elements in the general case, so for the decays of ^{130}Te and ^{128}Te we shall make use of the approximation (2.42) and fall back on the value 0.1 for M_2

and M_3.

We shall now examine the decays in greater detail.

$$\underline{{}^{130}Te_{52} \rightarrow {}^{130}Xe_{54}}$$

The decay scheme is shown in Figure 3. The energy release is 2.5 Mev ($\varepsilon_0 = 5$), but we do not know the actual value of δ. Using the estimate $M_2 = 0.1$, it follows from (4.6) that, if η is real, $\delta\left({}^{130}Te\right)$ must be greater than 2.1. The dependence of η on δ is shown graphically in Figure 4. If we accept the estimate[12,36] that $\delta < 20$, then equation (4.6) implies that

$$0.23 \times 10^{-4} < \eta < 1.25 \times 10^{-3} \tag{4.8}$$

This upper limit is somewhat higher than the estimate $\eta \approx 10^{-3}$ made by Primakoff and Rosen[31]. It should be pointed out, however, that this estimate is based on the assumption that the $\beta\beta$ decay of ${}^{130}Te$ is predominantly of the no-neutrino variety. We recall that the overall experimental rate of ${}^{130}Te$ $\beta\beta$ decay is $\lambda = 3.2 \times 10^{-22}$ year^{-1}. It can be seen from Figure 5, where we plot the theoretical two-neutrino rate $\lambda^{(2)}$ versus δ, that the assumption $\lambda^{(2)} \ll \lambda$ implies that δ is of the order of 20. It is difficult to reconcile this conclusion with the neglect, in Reference (31), of δ in comparison with k in the energy denominators $E_i + k + \delta$ $(i = 1, 2)$ in equation (2.12).

$$\underline{{}^{48}Ca_{20} \rightarrow {}^{48}Ti_{22}}$$

The decay scheme is shown in Figure 6. The energy release for this decay is 4.24 Mev ($\varepsilon_0 = 8.48$). We shall use the results of Bardin

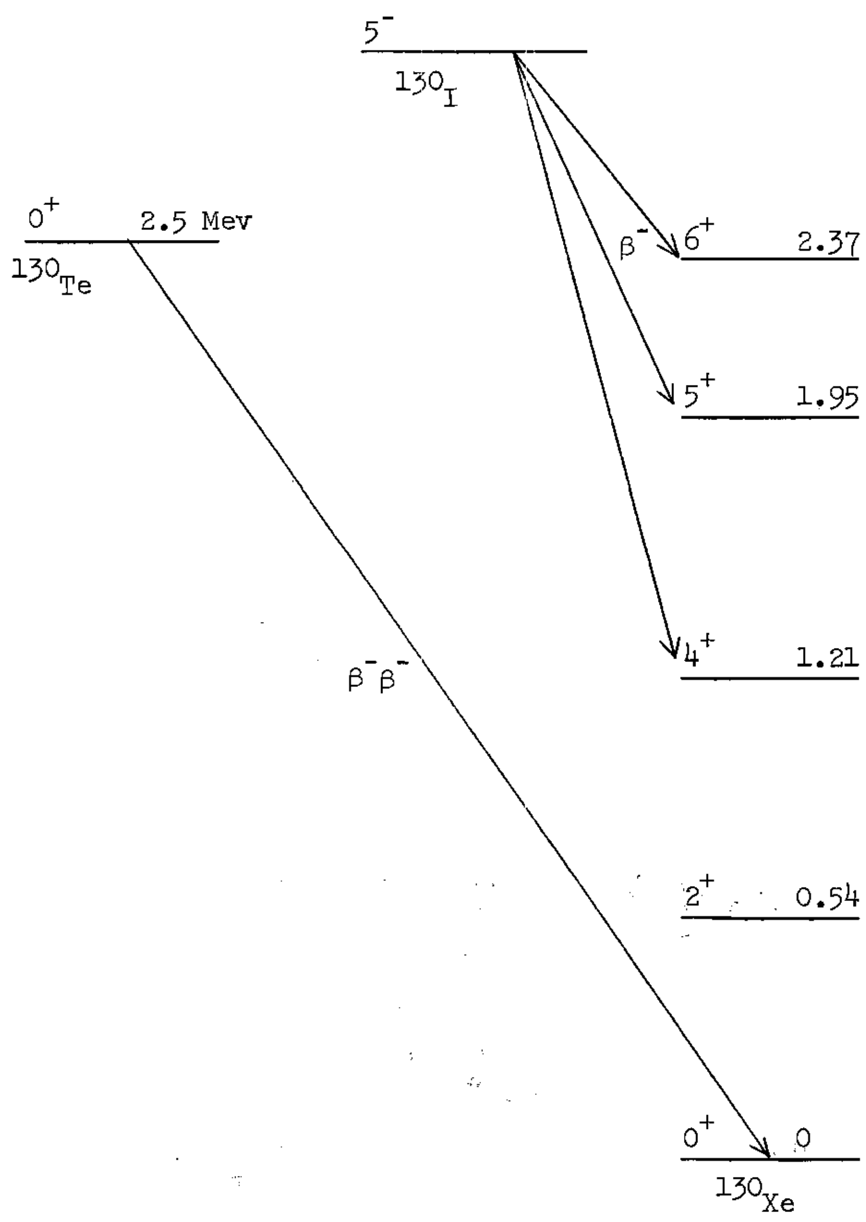

Figure 3. Decay Scheme of ^{130}Te.

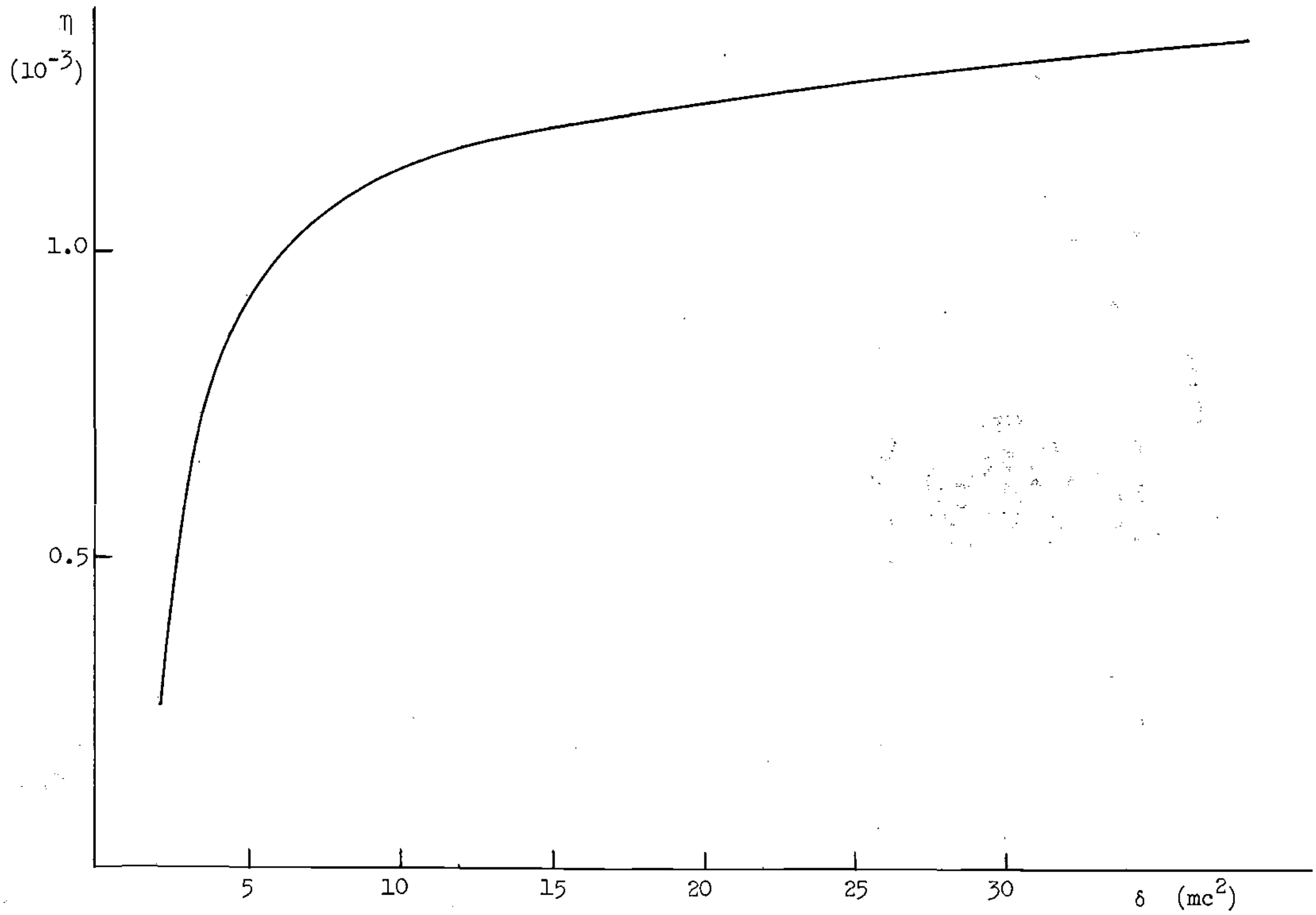

Figure 4. Graph of η vs $\delta(^{130}Te)$

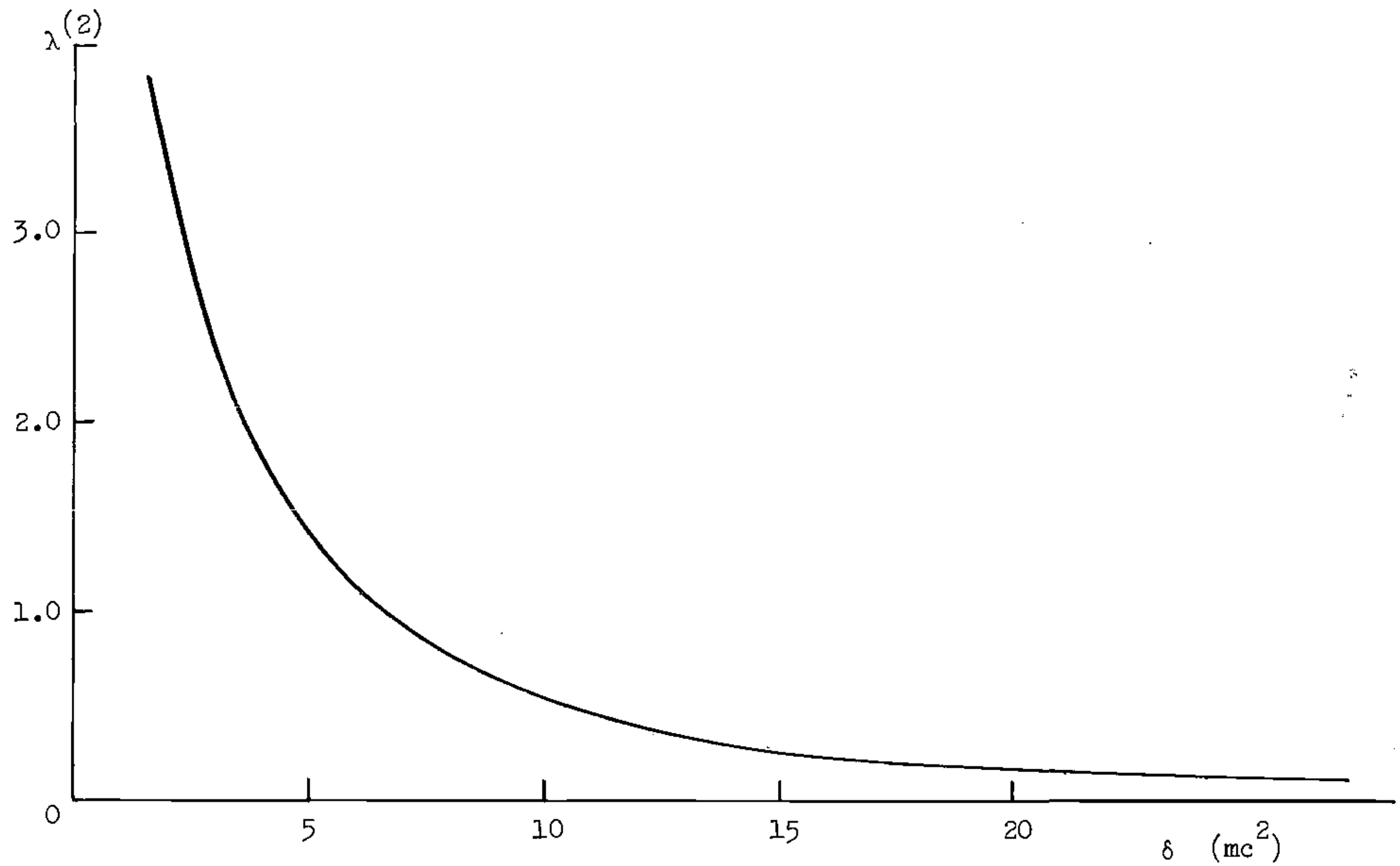

Figure 5. Two-neutrino decay rate $\lambda^{(2)}$ vs $\delta\left({}^{130}Te\right)$.
$\lambda^{(2)}$ is measured in units of 10^{-22} year^{-1}.

et al, viz,

$$\lambda^{(o)} < 0.35 \times 10^{-21} \text{ year}^{-1} \tag{4.9}$$

$$\lambda^{(2)} < 0.19 \times 10^{-19} \text{ year}^{-1} \tag{4.10}$$

and the estimates (4.7) of Khodel for the magnitudes of the matrix elements to obtain some information on the value of $\delta\left({}^{48}\text{Ca}\right)$ and the parameter η. Substituting $\epsilon_o = 8.48$ and $M_2 = 0.18$ into equation (3.40) (which gives an expression for the two-neutrino rate $\lambda^{(2)}$), we get

$$\delta\left({}^{48}\text{Ca}\right) > 2.85,$$

using (4.10). The expression (2.74) for the rate of the no-neutrino process is modified by the use of Khodel's estimates. Putting $M_3 = 0$ into (2.74), we get

$$\lambda^{(o)} = \frac{32}{81}\pi^6\eta^2\left(3g_A^2 + g_V^2\right)\left(\frac{m^7c^4}{h^{11}}G^4\right)\left(\frac{2\pi Z}{137}\right)^2\left[1 - \exp\left(-\frac{2\pi Z}{137}\right)\right]^{-2} \times$$

$$\times \left\{P'^2 f(\epsilon_o) + 4\,Q'^2 g(\epsilon_o) - 4\,P'Q'h(\epsilon_o)\right\} \left|M_2\right|^2 \tag{4.11}$$

where
$$P' = \frac{3x-1}{R} - 2\left(\frac{mc}{h}\right)\left(\epsilon_o + 2\delta + 2\right) \tag{4.12}$$

$$Q' = \frac{1}{R} - 2\left(\frac{mc}{h}\right)\left(\epsilon_o + 2\delta + 2\right) \tag{4.13}$$

$$x = \frac{3g_A^2 - g_V^2}{3g_A^2 + g_V^2} \tag{4.14}$$

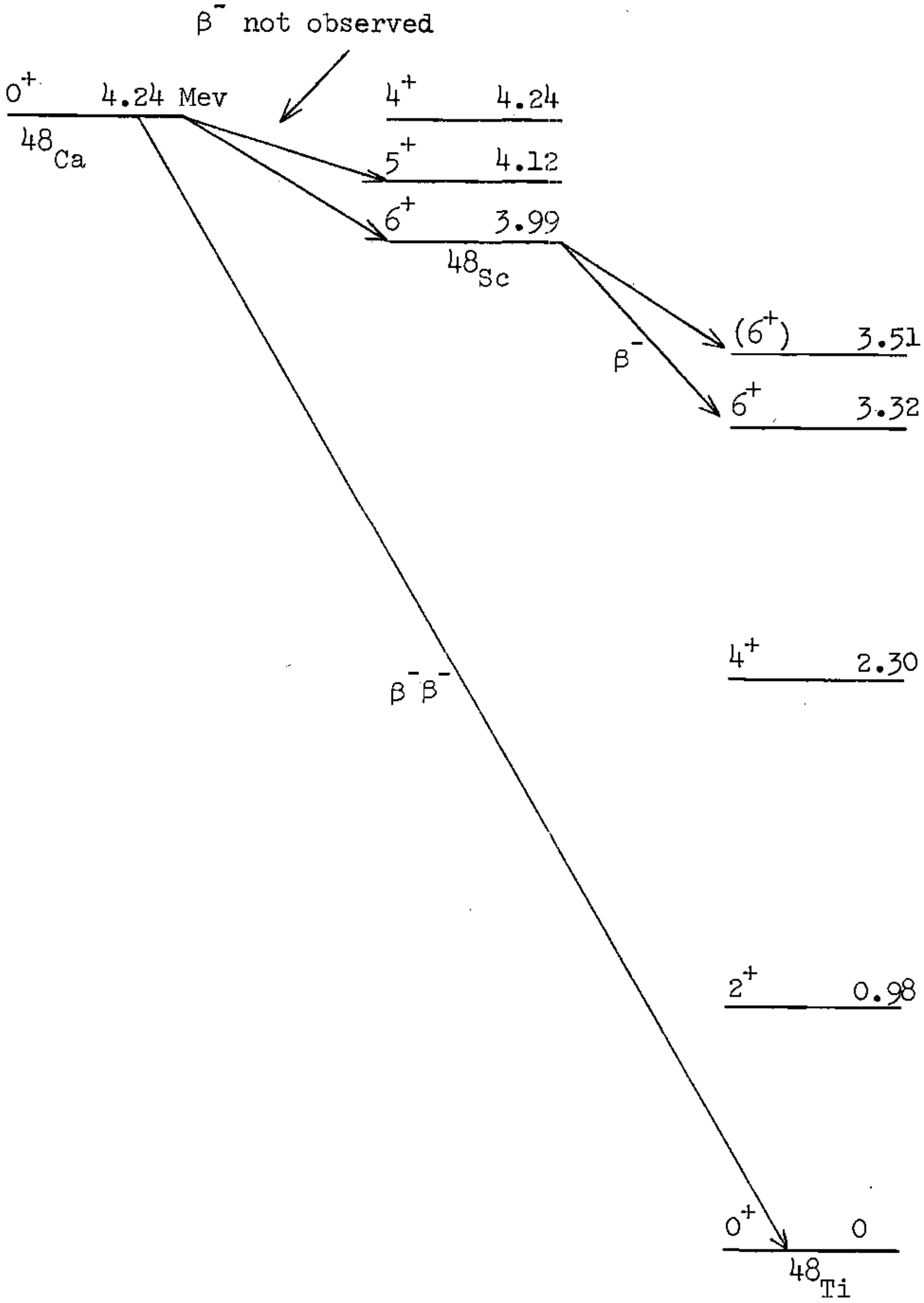

Figure 6. Decay Scheme of ^{48}Ca

From (4.11) we obtain

$$\eta^2 = \frac{81}{32}\left(\frac{137}{2\pi Z}\right)^2 h^{11} \times \frac{\left[1 - \exp\left(-\frac{2\pi Z}{137}\right)\right]^2}{G^4 \pi^6 m^7 c^4} \times \frac{\ln 2}{3.156 \times 10^7 \times 10^{T_o} \mid M_2\mid^2} \times$$

$$\times \frac{1}{(3g_A^2 + g_V^2)^2 \left[P'^2 f(\epsilon_o) + 4\, Q'^2 g(\epsilon_o) - 4\, P'Q' h(\epsilon_o)\right]} \qquad (4.15)$$

whence $$\eta < 0.53 \times 10^{-3} \qquad (4.16)$$

using $T_o > 21.3$ and $\delta > 2.85$.

<u>$^{128}Te_{52} \rightarrow {}^{128}Xe_{54}$</u>

The decay scheme is shown in Figure 7. The energy release for this decay is 0.85 Mev ($\epsilon_o = 1.7$). Assuming that the 1^+ state of the intermediate nucleus ^{128}I is the only one which contributes (in the allowed approximation) to the decay rate, or that the energy difference between a possible spin zero state and the 1^+ state is small, we have $\delta = 2.5$. Since we are assuming that η is real, we may obtain an immediate upper limit to the value of T from equation (4.6) by inserting the values $\epsilon_o = 1.7$, $\delta = 2.5$ and $Z = 54$. This gives

$$T\left({}^{128}Te\right) < 24.95 \qquad (4.17)$$

We may also use (4.6) to plot a graph of $T\left({}^{128}Te\right)$ versus η, and this is shown in Figure 8. Using the inequality

$$0.23 \times 10^{-4} < \eta < 1.25 \times 10^{-3} \qquad (4.8)$$

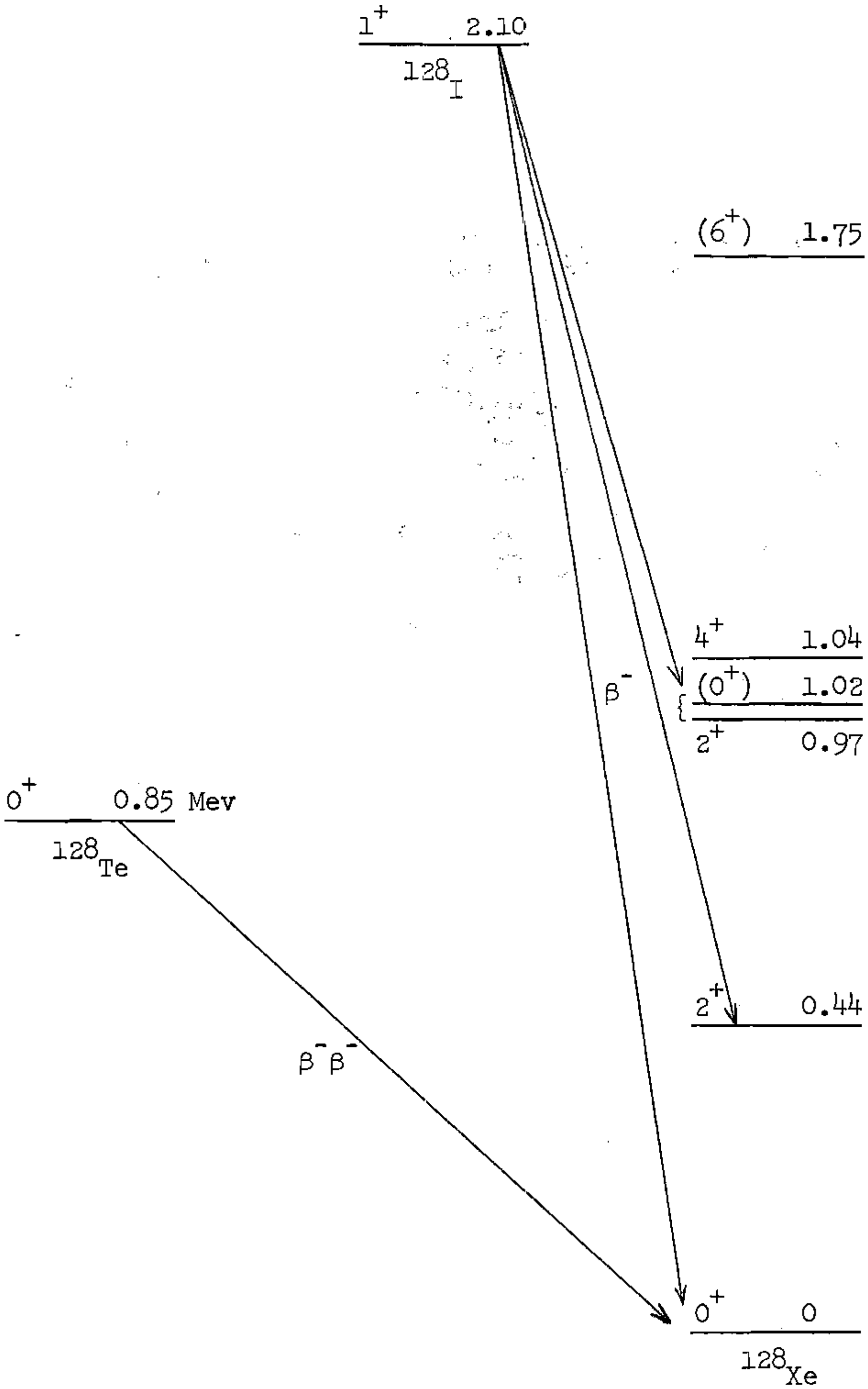

Figure 7. Decay Scheme of ^{128}Te.

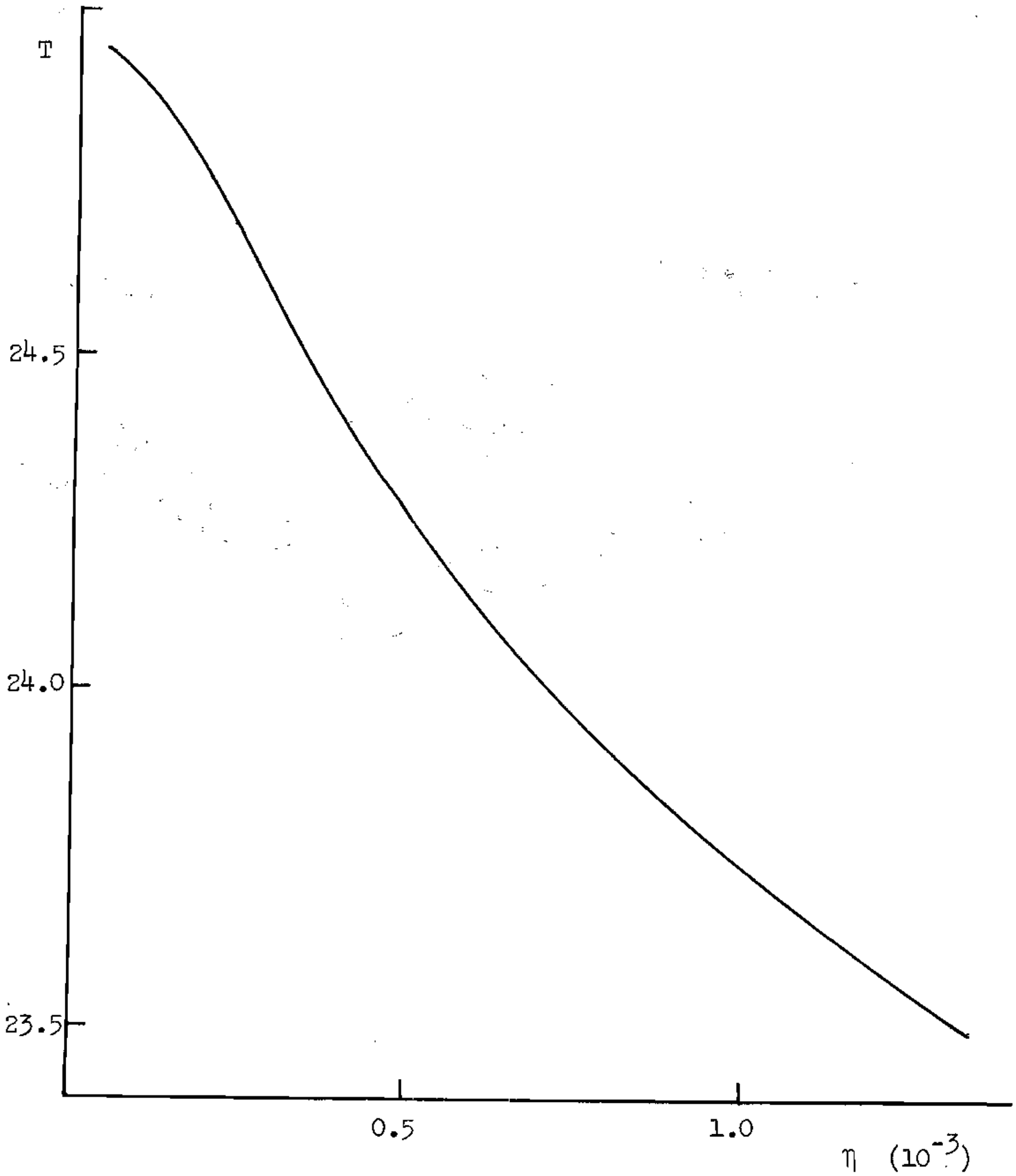

Figure 8. Graph of $T(^{128}Te)$ vs η

we may infer that $T > 23.56$, and we have

$$23.56 < T < 24.95 \tag{4.18}$$

On the other hand, the upper limit

$$\eta < 0.53 \times 10^{-3} \tag{4.16}$$

yields the inequality

$$24.24 < T < 24.95 \tag{4.19}$$

The lower limits on T in (4.18) and (4.19) give values for the half-life of the ^{128}Te $\beta\beta$ decay which are somewhat higher than the experimental value of $10^{22.5}$ years obtained by Takaoka and Ogata[16]. This is not surprising, in view of the reservations expressed by these authors, viz, that the excess of ^{128}Xe which they found may not be due entirely to ^{128}Te $\beta\beta$ decay.

CHAPTER V

CONCLUSIONS

The experimental and theoretical effort devoted in recent years to the study of $\beta\beta$ decay processes has been stimulated in large part by an interest in determining whether the law of lepton number conservation is violated in weak interactions and, if so, in estimating the magnitude of the lepton non-conservation parameter η. It must be emphasized that the only way to settle this question unambiguously is to observe the electron sum energy spectrum. In spite of many attempts, this has still not been done satisfactorily.

On the theoretical side, efforts to give a complete picture of $\beta\beta$ decay have been hampered by the difficulty of calculating the magnitudes of the nuclear matrix elements involved. Some progress has been made in this direction by Khodel[34], who has considered the simplest case in which the initial nucleus, close to magic, has only a pair of neutrons outside closed shells. These neutrons, at the single particle level ν_1, are transformed into a pair of protons having the same angular momentum I, at level ν_2. However, most of the nuclei which undergo $\beta\beta$ decay do not meet these requirements and the situation is considerably more complicated. In these cases, the traditional method of estimating the magnitudes of the nuclear matrix elements has been used.

The calculation of the $\beta\beta$ decay rate can be greatly simplified if account is taken of the fact that the maximum value of the momentum of the virtual neutrino is of the order of the reciprocal of the distance between the nucleons, i.e. those momenta k which predominate in the integration over the neutrino momenta are much larger than the remaining terms in the energy denominators $E_i + k + \delta$ ($i = 1, 2$) of equation (2.12). Some authors(31,36) have used approximations in which the energy gap $\delta \equiv \langle E_N \rangle - E_i$ does not appear. We have chosen, instead, to expand the reciprocal of the denominator in powers of $\frac{E_i + k}{k}$, keeping only the linear term. In this way we have been able to observe the dependence of the decay rates on the quantity δ. This shows that the branching ratio $\lambda^{(o)}/(\lambda^{(o)} + \lambda^{(2)})$, and hence the parameter η, is quite sensitive to the magnitude of δ. In this regard, the assumption of Primakoff and Rosen(31) that the $\beta\beta$ decay of ^{130}Te is predominantly of the no-neutrino variety may be open to question.

The experimental data on the three decays investigated falls into three categories. In each case the energy release ϵ_o is known.

(i) $^{130}Te \rightarrow {}^{130}Xe$.

The half-life $\tau = 10^T$ years is known, but there is no experimental information on the magnitude of δ.

(ii) $^{48}Ca \rightarrow {}^{48}Ti$.

Upper limits have been set on the no-neutrino rate $\lambda^{(o)}$ and the two-neutrino rate $\lambda^{(2)}$, but no experimental information on the magnitude of δ is available.

(iii) $^{128}Te \rightarrow {}^{128}Xe$.

The half-life of this decay is known, but with a possibly large experimental error. The energy level of the 1^+ intermediate state is known, and it has been assumed that a possible 0^+ intermediate state is close to the 1^+ state, giving an estimate $\delta = 2.5$.

We have substituted the data in (i) above into the expression (4.6) for η^2 and, using the requirement that η be real, we have obtained the lower limit

$$\delta\left({}^{130}\mathrm{Te}\right) > 2.1$$

The estimate of Primakoff and Rosen[12] that $\delta < 20$ yields the relation

$$0.23 \times 10^{-4} < \eta < 1.25 \times 10^{-3}$$

The data in (ii) has then been used to obtain the limits

$$\delta\left({}^{48}\mathrm{Ca}\right) > 2.85$$

$$\eta < 0.53 \times 10^{-3}$$

Turning to (iii), we use the estimate $\delta\left({}^{128}\mathrm{Te}\right) = 2.5$ and the fact that η is real to obtain the upper limit

$$T\left({}^{128}\mathrm{Te}\right) < 24.95$$

The upper limit on η can be used to obtain a lower limit on $T\left({}^{128}\mathrm{Te}\right)$ from equation (4.6) or, alternatively, from Figure 8. If we use the limit $\eta < 1.25 \times 10^{-3}$, obtained from (i), we get $T > 23.56$. On the other hand, the limit $\eta < 0.53 \times 10^{-3}$ obtained from (ii) yields $T > 24.24$.

It should be recalled that the estimate on which the upper limit $\eta < 1.25 \times 10^{-3}$ is based is that δ is no greater than 20. It is, of course, conceivable that δ is considerably smaller than this figure, as Primakoff and Rosen have pointed out. Indeed, the limit $\eta < 0.53 \times 10^{-3}$ obtained from the experimental results of Bardin et al. indicates (see

Figure 4) that $\delta\left({}^{130}\mathrm{Te}\right) < 2.68$. Admittedly, only one event was seen by these authors which could definitely be construed as neutrinoless decay. We are nonetheless inclined to accept the limit on η based on these experiments, and to conclude that

$$0.23 \times 10^{-4} < \eta < 0.53 \times 10^{-3}$$

and hence that

$$24.24 < T\left({}^{128}\mathrm{Te}\right) < 24.95.$$

APPENDIX I

EVALUATION OF SOME INTEGRALS

In this Appendix we evaluate some of the integrals which are needed for the evaluation of the matrix element of the no-neutrino process.

(i)
$$\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{e^{-i\vec{k}\,\vec{r}}}{k^2} = \frac{2\pi}{(2\pi)^3} \int_0^\infty \frac{dk}{ikr} \int_0^\pi e^{-ikr\cos\theta} (ik\sin\theta\, d\theta)$$

$$= \frac{1}{(2\pi)^2} \frac{1}{ir} \int_0^\infty \frac{dk}{k} \left[e^{-ikr\cos\theta} \right]_0^\pi$$

$$= \frac{2}{(2\pi)^2 r} \int_0^\infty \frac{dk}{k} \left(\frac{e^{ikr} - e^{-ikr}}{2i} \right)$$

$$= \frac{1}{2\pi^2 r} \int_0^\infty \frac{\sin kr}{k}\, dk$$

and using
$$\int_0^\infty \frac{\sin mx}{x}\, dx = \frac{\pi}{2}\, ; \quad m > 0$$

we get

$$\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{e^{-i\vec{k}\cdot\vec{r}}}{k^2} = \frac{1}{4\pi r} \tag{I.1}$$

(ii) Let
$$I = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{\vec{V}\cdot\vec{k}}{k^n} e^{-i\vec{k}\cdot\vec{r}} \quad ; \quad n = 1, 2, 3 \ldots$$

where
$$\vec{V} = \vec{\ell}_{\lambda\mu}$$

and let
$$\vec{r} = (r, 0, 0)$$
$$\vec{V} = (V \sin\alpha \cos\beta, V \sin\alpha \sin\beta, V \cos\alpha)$$
$$\vec{k} = (k \sin\theta \cos\varphi, k \sin\theta \sin\varphi, k \cos\varphi)$$

i.e. the vector r lies in the polar direction and the vectors $\vec{V}$ and $\vec{k}$ have spherical polar coordinates (V, α, β) and (k, θ, φ) respectively. Then

$$I = \frac{V}{(2\pi)^3} \int_0^\infty dk \frac{k^3}{k^n} \int_{-1}^{1} d\mu\, e^{-ikr\mu} \int d\varphi \left[\cos\theta \cos\alpha + \sin\theta \sin\alpha \cos(\beta - \varphi)\right]$$

$$= \frac{2\pi V \cos\alpha}{(2\pi)^3} \int dk \frac{k^3}{k^n} \int_{-1}^{1} \mu\, e^{-ikr\mu} d\mu \quad ; \quad \mu = \cos\theta \qquad (I.2)$$

Thus
$$I = \frac{V \cos\alpha}{(2\pi)^2} \int_0^\infty dk \frac{k^3}{k^n} \left(-\frac{1}{ir}\right) \frac{d}{dk} \int_{-1}^{1} e^{-ikr\mu} d\mu$$

$$= \frac{V \cos\alpha}{(2\pi)^2} \int_0^\infty dk \frac{k^3}{k^n} \left(-\frac{1}{ir}\right) \frac{d}{dk} \left(\frac{2}{kr} \sin kr\right)$$

$$= \frac{1}{i} \frac{\vec{V}\cdot\vec{r}}{2\pi^2 r^2} \int_0^\infty dk \frac{k^2}{k^n} \left(\frac{\sin kr}{kr} - \cos kr\right) \qquad (I.3)$$

where $\vec{V}\cdot\vec{r} = Vr\cos\alpha$. If $n = 2$, we get

$$\vec{V}\cdot\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{\vec{k}}{k^2}e^{-i\vec{k}\cdot\vec{r}} = \frac{1}{i}\frac{\vec{V}\cdot\vec{r}}{2\pi^2r^2}\int_0^\infty\left(\frac{\sin kr}{kr} - \cos kr\right)dk \tag{I.4}$$

On carrying out the integration, we get

$$\vec{V}\cdot\int\frac{d^3k}{(2\pi)^3}\frac{\vec{k}}{k^2}e^{-i\vec{k}\cdot\vec{r}} = \vec{V}\cdot\int\frac{d^3k}{(2\pi)^3}\frac{\vec{k}}{k^2}e^{+i\vec{k}\cdot\vec{r}}$$

$$= \frac{\vec{V}\cdot\vec{r}}{4\pi i r^3} \tag{I.5}$$

If $n = 3$, we get

$$\vec{V}\cdot\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{\vec{k}}{k^3}e^{-i\vec{k}\cdot\vec{r}} = \frac{1}{i}\frac{\vec{V}\cdot\vec{r}}{2\pi^2r^2}\int_0^\infty\frac{dk}{k}\left(\frac{\sin kr}{kr} - \cos kr\right) \tag{I.6}$$

Now $\displaystyle\int_0^\infty\frac{dk}{k}\left(\frac{\sin kr}{kr} - \cos kr\right) = \lim_{t\to 0}\int_t^\infty\left(\frac{\sin x}{x^2} - \frac{\cos x}{x}\right)dx$

and using the result

$$\int\frac{\sin x}{x^m}dx = -\frac{\sin x}{(m-1)x^{m-1}} + \frac{1}{m-1}\int\frac{\cos x}{x^{m-1}}dx \quad ; \quad m = 2$$

we get $\displaystyle\vec{V}\cdot\int\frac{d^3k}{(2\pi)^3}\frac{\vec{k}}{k^3}e^{-i\vec{k}\cdot\vec{r}} = -\vec{V}\cdot\int\frac{d^3k}{(2\pi)^3}\frac{\vec{k}}{k^3}e^{+i\vec{k}\cdot\vec{r}}$

$$= \frac{1}{i}\frac{\vec{V}\cdot\vec{r}}{2\pi^2r^2}\lim_{t\to 0}\left[-\frac{\sin x}{x}\right]_t^\infty = \frac{1}{i}\frac{\vec{V}\cdot\vec{r}}{2\pi^2r^2} \tag{I.7}$$

APPENDIX II

EXTRACTION OF SCALAR OPERATORS FROM NUCLEAR MATRIX ELEMENTS

In Chapter II we derived the following expression for the matrix element for neutrinoless $\beta\beta$ decay :

$$M = \frac{G^2}{16\pi}(E_1 - E_2)\ell_{\lambda\mu} \langle N_f|2\sum_{n,m} \tau_n^{(+)}\tau_m^{(+)}(\Gamma_\lambda)_n(\Gamma_\mu)_m \frac{e^{-i(\vec{p}_1\cdot\vec{r}_n + \vec{p}_2\cdot\vec{r}_m)}}{|\vec{r}_n - \vec{r}_m|}|N_i\rangle -$$

$$- \frac{G^2}{8\pi}\vec{\ell}\,'_{\lambda\mu} \langle N_f|2\sum_{n,m} \tau_n^{(+)}\tau_m^{(+)}(\Gamma_\lambda)_n(\Gamma_\mu)_m \frac{e^{-i(\vec{p}_1\cdot\vec{r}_n + \vec{p}_2\cdot\vec{r}_m)}}{|\vec{r}_n - \vec{r}_m|^3}(\vec{r}_n - \vec{r}_m)|N_i\rangle +$$

$$+ \frac{G^2}{8\pi^2}(E_1 + E_2 + 2\delta)\vec{\ell}\,'_{\lambda\mu}\langle N_f|2\sum_{n,m} \tau_n^{(+)}\tau_m^{(+)}(\Gamma_\lambda)_n(\Gamma_\mu)_m \frac{e^{-i(\vec{p}_1\cdot\vec{r}_n + \vec{p}_2\cdot\vec{r}_m)}}{|\vec{r}_n - \vec{r}_m|^2}(\vec{r}_n - \vec{r}_m)|N_i\rangle \tag{2.26}$$

It is our task in this Appendix to extract from the operators sandwiched between the nuclear states those parts which are invariant under rotations and inversions of the nuclear coordinates, as required for $0^+ \rightarrow 0^+$ transitions. The isospin operators $\tau_n^{(+)}$ and $\tau_m^{(+)}$ and powers of $|\vec{r}_n - \vec{r}_m|$ appearing in the denominators of (2.26) will be omitted where these play no essential role in the ensuing discussion. We will therefore focus our attention on the two expressions

$$\ell_{\lambda\mu} \langle N_f|2\sum_{n,m} \ldots (\Gamma_\lambda)_n(\Gamma_\mu)_m\, e^{-i(\vec{p}_1\cdot\vec{r}_n + \vec{p}_2\cdot\vec{r}_m)}|N_i\rangle \tag{II.1}$$

$$\vec{\ell}'_{\lambda\mu} \cdot \langle N_f | 2 \sum_{n,m} \ldots (\Gamma_\lambda)_n (\Gamma_\mu)_m \, e^{-i(\vec{p}_1\cdot\vec{r}_n + \vec{p}_2\cdot\vec{r}_m)} (\vec{r}_n - \vec{r}_m) | N_i \rangle \qquad \text{(II.2)}$$

The expression (II.2) may be written as

$$\ell_{\lambda\mu} \langle N_f | \sum_{n,m} \ldots (\Gamma_\lambda)_n (\Gamma_\mu)_m \left\{ e^{-i(\vec{p}_1\cdot\vec{r}_n + \vec{p}_2\cdot\vec{r}_m)} + e^{-i(\vec{p}_1\cdot\vec{r}_m + \vec{p}_2\cdot\vec{r}_n)} \right\} | N_i \rangle \; +$$

$$\text{(II.3)}$$

$$+ \; \ell_{\lambda\mu} \langle N_f | \sum_{n,m} \ldots (\Gamma_\lambda)_n (\Gamma_\mu)_m \left\{ e^{-i(p_1 r_n + p_2 r_m)} - e^{-i(p_1 r_m + p_2 r_n)} \right\} | N_i \rangle$$

Keeping only the first two terms of the exponentials, we obtain an expression which may be written symbolically as

$$\ell_{\lambda\mu} (\Gamma_\lambda)_n (\Gamma_\mu)_m \left[2 - i(\vec{p}_1 + \vec{p}_2)\cdot(\vec{r}_n + \vec{r}_m) - (\vec{p}_1 - \vec{p}_2)\cdot(\vec{r}_n - \vec{r}_m) \right] \qquad \text{(II.4)}$$

summation over the indices n and m being always understood. We shall consider each of the terms in (II.4) separately.

Let $\quad X_1 = 2\ell_{\lambda\mu}(\Gamma_\lambda)_n(\Gamma_\mu)_m \equiv \ell_{\lambda\mu} \langle N_f | 2 \sum_{n,m} \ldots (\Gamma_\lambda)_n (\Gamma_\mu)_m | N_i \rangle$

$$= \ell_{\lambda\mu} \left[(\Gamma_\lambda)_n (\Gamma_\mu)_m + (\Gamma_\mu)_n (\Gamma_\lambda)_m \right] \qquad \text{(II.5)}$$

It should be noted that the operators $(\Gamma_\lambda)_n$ and $(\Gamma_\mu)_m$ commute, since they operate in different spaces. Now the expression in square brackets is symmetric in the indices λ and μ, and hence we need only consider the symmetric part of $\ell_{\lambda\mu}$, which gives

$$X_1 = \frac{1}{2}\left(\ell_{\lambda\mu} + \ell_{\mu\lambda}\right)\left[(\Gamma_\lambda)_n(\Gamma_\mu)_m + (\Gamma_\mu)_n(\Gamma_\lambda)_m\right] \tag{II.6}$$

where $$\ell_{\lambda\mu} \equiv (2\eta)\bar{u}(\vec{p}_1)\gamma_\lambda\gamma_4\gamma_\mu u^*(\vec{p}_2) \tag{2.8}$$

Equation (2.8) may be written symbolically as

$$\ell_{\lambda\mu} = (2\eta)\ \gamma_\lambda\gamma_4\gamma_\mu$$

Thus we get

$$X_1 = \frac{1}{2}(2\eta)(\gamma_\lambda\gamma_4\gamma_\mu + \gamma_\mu\gamma_4\gamma_\lambda)\left[(\Gamma_\lambda)_n(\Gamma_\mu)_m + (\Gamma_\mu)_n(\Gamma_\lambda)_m\right] \tag{II.7}$$

$$= (2\eta)\left(\gamma_\lambda\delta_{\mu4} + \gamma_\mu\delta_{\lambda4} - \gamma_4\delta_{\lambda\mu}\right)\left[(\Gamma_\lambda)_n(\Gamma_\mu)_m + (\Gamma_\mu)_n(\Gamma_\lambda)_m\right] \tag{II.8}$$

using the anti-commutation relation

$$\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha = 2\ \delta_{\alpha\beta}$$

It is clear from the expression (2.27) for $(\Gamma_\lambda)_n(\Gamma_\mu)_m$ that we need only consider the cases $\lambda = 4$; $\mu = 4$ and $\lambda = j$; $\mu = k$. In the first case (II.8) reduces to

$$4\eta\ \gamma_4 g_V^2 \equiv 4\eta\ \bar{u}(\vec{p}_1)\gamma_4 u^*(\vec{p}_2)\ \langle N_f| \sum_{n,m} \cdots\ g_V^2 |N_i\rangle \tag{II.9}$$

In the second case ($\lambda = j$; $\mu = k$) we get

$$-(2\eta)\delta_{jk}\gamma_4(-g_A^2)(\sigma_j\sigma_k + \sigma_k\sigma_j) = 4\eta\ g_A^2\ \gamma_4\ \vec{\sigma}\cdot\vec{\sigma}$$

$$\equiv 4\eta\ g_A^2\ \bar{u}(\vec{p}_1)\gamma_4 u^*(\vec{p}_2)\ \langle N_f| \sum_{n,m} \cdots\ \vec{\sigma}_n\cdot\vec{\sigma}_m |N_i\rangle \tag{II.10}$$

Thus from (II.9) and (II.10) we see that

$$X_1 = 4\eta\ \bar{u}(\vec{p}_1)\gamma_4 u^*(\vec{p}_2)\Big\{g_V^2\langle N_f|\sum_{n,m}\tau_n^{(+)}\tau_m^{(+)}|N_i\rangle + g_A^2\langle N_f|\sum_{n,m}\tau_n^{(+)}\tau_m^{(+)}\vec{\sigma}_n\cdot\vec{\sigma}_m|N_i\rangle\Big\} \tag{II.11}$$

which clearly satisfies the requirements of parity and rotation invariance.

We now consider the term

$$X_2 = -\,i\ \ell_{\lambda\mu}(\Gamma_\lambda)_n(\Gamma_\mu)_m(\vec{p}_1+\vec{p}_2)\cdot(\vec{r}_n+\vec{r}_m) \tag{II.12}$$

$$= -\frac{i}{2}\ \ell_{\lambda\mu}\Big[(\Gamma_\lambda)_n(\Gamma_\mu)_m + (\Gamma_\mu)_n(\Gamma_\lambda)_m\Big]\ \vec{P}\cdot\vec{R} \tag{II.13}$$

where $\vec{P} \equiv \vec{p}_1 + \vec{p}_2$; $\vec{R} \equiv \vec{r}_n + \vec{r}_m$

Again taking only the symmetric part of $\ell_{\lambda\mu}$, we get

$$X_2 = -\frac{i}{2}\eta(\gamma_\lambda\gamma_4\gamma_\mu + \gamma_\mu\gamma_4\gamma_\lambda)\Big[(\Gamma_\lambda)_n(\Gamma_\mu)_m + (\Gamma_\mu)_n(\Gamma_\lambda)_m\Big]\ P_a R_a \tag{II.14}$$

$$= -\,i\eta(\gamma_\lambda\delta_{\mu 4} + \gamma_\mu\delta_{\lambda 4} - \gamma_4\delta_{\lambda\mu})\Big[(\Gamma_\lambda)_n(\Gamma_\mu)_m + (\Gamma_\mu)_n(\Gamma_\lambda)_m\Big]\ P_a R_a \tag{II.15}$$

Putting $\lambda = 4$; $\mu = 4$ into (II.15) yields

$$-\,i(2\eta)\gamma_4 P_a R_a g_V^2 \equiv -\,i(2\eta)g_V^2\,\bar{u}(\vec{p}_1)\gamma_4 u^*(\vec{p}_2)\ \vec{P}\cdot\langle N_f|\sum_{n,m}\tau_n^{(+)}\tau_m^{(+)}(\vec{r}_n+\vec{r}_m)|N_i\rangle$$

and a scalar clearly cannot be formed from the nucleon operators. Putting $\lambda = j$; $\mu = k$ into (II.15) yields

$$-\,i\eta(-\,g_A^2)(-\,\gamma_4)(\sigma_j\sigma_k + \sigma_k\sigma_j)\ P_a R_a$$

which we write explicitly as

$$-\,i\eta\, g_A^2\, \bar{u}(p_1)\gamma_4 u^*(p_2)\ \vec{P}\,\langle N_f|\sum_{n,m}\tau_n^{(+)}\tau_m^{(+)}(\vec{\sigma}_n\cdot\vec{\sigma}_m)(\vec{r}_n+\vec{r}_m)|N_i\rangle$$

Again there is no nucleon scalar available.

Finally, we consider the term

$$X_3 = \mp i\, \ell_{\lambda\mu}(\Gamma_\lambda)_n(\Gamma_\mu)_m(\vec{p}_1-\vec{p}_2)\cdot(\vec{r}_n-\vec{r}_m) \qquad \text{(II.16)}$$

$$= -\frac{i}{2}\,\ell_{\lambda\mu}\left[(\Gamma_\lambda)_n(\Gamma_\mu)_m-(\Gamma_\mu)_n(\Gamma_\lambda)_m\right]\vec{p}\cdot\vec{r} \qquad \text{(II.17)}$$

where $\quad \vec{p} \equiv \vec{p}_1-\vec{p}_2\ ;\ \vec{r} \equiv \vec{r}_n-\vec{r}_m$

Since the expression in square brackets in (II.17) is antisymmetric in the indices λ and μ, $\ell_{\lambda\mu}$ will be replaced by its antisymmetric part, viz,

$$\frac{1}{2}(\ell_{\lambda\mu}+\ell_{\mu\lambda}) = \eta(\gamma_\lambda\gamma_4\gamma_\mu-\gamma_\mu\gamma_4\gamma_\lambda)$$

$$= \eta\left[\gamma_\lambda(2\delta_{\mu4}-\gamma_\mu\gamma_4)-\gamma_\mu(2\delta_{\lambda4}-\gamma_\lambda\gamma_4)\right]$$

$$= 2\eta\left[\gamma_\lambda\delta_{\mu4}-\gamma_\mu\delta_{\lambda4}-\frac{1}{2}(\gamma_\lambda\gamma_\mu-\gamma_\mu\gamma_\lambda)\gamma_4\right]$$

$$= 2\eta\left[\gamma_\lambda\delta_{\mu4}-\gamma_\mu\delta_{\lambda4}-\gamma_\lambda\gamma_\mu\gamma_4\right] \qquad \text{(II.18)}$$

since (II.17) vanishes for $\lambda = \mu$. Thus we obtain

$$X_3 = -i\eta\left(\gamma_\lambda\delta_{\mu4}-\gamma_\mu\delta_{\lambda4}-\gamma_\lambda\gamma_\mu\gamma_4\right)\left[(\Gamma_\lambda)_n(\Gamma_\mu)_m-(\Gamma_\mu)_n(\Gamma_\lambda)_m\right]p_a r_a \qquad \text{(II.19)}$$

Putting $\lambda = j$; $\mu = k$ into (II.19), we get

$$X_3 = - ig_A^2\eta(\gamma_j\gamma_k\gamma_4)(\sigma_j\sigma_k - \sigma_k\sigma_j)\ p_a r_a$$

$$= - ig_A^2\eta\ A_{jk}\ e_{tjk}\ (\vec{\sigma}_n \times \vec{\sigma}_m)_t\ (p_a r_a) \qquad \text{(II.20)}$$

where
$$A_{jk} \equiv \bar{u}(p_1)\gamma_j\gamma_k\gamma_4 u^*(p_2) \qquad \text{(II.21)}$$

and e_{tjk} is the totally antisymmetric unit third rank tensor. It is clear from (II.20) that no scalar can be formed from the nucleon operators.

We now turn our attention to the expression (II.2). This may be written as

$$\vec{\ell}'_{\lambda\mu}\cdot\langle N_f|\sum_{n,m}\ldots(\Gamma_\lambda)_n(\Gamma_\mu)_m\left\{e^{-i(\vec{p}_1\cdot\vec{r}_n+\vec{p}_2\cdot\vec{r}_m)} + e^{-i(\vec{p}_1\cdot\vec{r}_m+\vec{p}_2\cdot\vec{r}_n)}\right\}(\vec{r}_n-\vec{r}_m)|N_i\rangle$$

$$\text{(II.22)}$$

$$+\ \vec{\ell}'_{\lambda\mu}\cdot\langle N_f|\sum_{n,m}\ldots(\Gamma_\lambda)_n(\Gamma_\mu)_m\left\{e^{-i(\vec{p}_1\cdot\vec{r}_n+\vec{p}_2\cdot\vec{r}_m} - e^{-i(\vec{p}_1\cdot\vec{r}_m+\vec{p}_2\cdot\vec{r}_n)}\right\}(\vec{r}_n-\vec{r}_m)|N_i\rangle$$

Keeping only the first two terms of the exponentials, we get

$$\vec{\ell}'_{\lambda\mu}\cdot\langle N_f|\sum_{n,m}\ldots(\Gamma_\lambda)_n(\Gamma_\mu)_m\left\{2 - i(\vec{p}_1+\vec{p}_2)\cdot(\vec{r}_n+\vec{r}_m)\right\}(\vec{r}_n-\vec{r}_m)|N_i\rangle\ -$$

$$- i\ \vec{\ell}'_{\lambda\mu}\cdot\langle N_f|\sum_{n,m}\ldots(\Gamma_\lambda)_n(\Gamma_\mu)_m\left\{(\vec{p}_1-\vec{p}_2)\cdot(\vec{r}_n-\vec{r}_m)\right\}(\vec{r}_n-\vec{r}_m)|N_i\rangle \qquad \text{(II.23)}$$

where
$$\vec{\ell}'_{\lambda\mu} \equiv -(2\eta)\bar{u}(\vec{p}_1)\gamma_\lambda\vec{\gamma}\ \gamma_\mu u^*(\vec{p}_2) \qquad \text{(2.9)}$$

Each term in the expression (II.23) will now be considered separately.

The first term is

$$Y_1 = \vec{\ell}'_{\lambda\mu}\cdot\langle N_f|2\sum_{n,m}\cdots(\Gamma_\lambda)_n(\Gamma_\mu)_m(\vec{r}_n - \vec{r}_m)|N_i\rangle \tag{II.24}$$

$$= \vec{\ell}'_{\lambda\mu}\cdot\langle N_f|\sum_{n,m}\cdots\left[(\Gamma_\lambda)_n(\Gamma_\mu)_m - (\Gamma_\mu)_n(\Gamma_\lambda)_m\right](\vec{r}_n - \vec{r}_m)|N_i\rangle \tag{II.25}$$

which we write symbolically as

$$Y_1 = -(2\pi)\, A_{\lambda i\mu}\left[(\Gamma_\lambda)_n(\Gamma_\mu)_m - (\Gamma_\mu)_n(\Gamma_\lambda)_m\right] r_i \tag{II.26}$$

where
$$A_{\lambda i\mu} \equiv \bar{u}(\vec{p}_1)\gamma_\lambda\gamma_i\gamma_\mu u^*(\vec{p}_2) \tag{II.27}$$

$$\vec{r} \equiv \vec{r}_n - \vec{r}_m$$

The expression in square brackets in (II.26) clearly vanishes for $\lambda = \mu$, and in particular for $\lambda = \mu = 4$, so we need only consider the case $\lambda = j$; $\mu = k$ $(j \neq k)$, which yields

$$Y_1 = 2g_A^2\pi\, A_{jik}\,(\sigma_j\sigma_k - \sigma_k\sigma_j)\, r_i \tag{II.28}$$

Since $(\sigma_j\sigma_k - \sigma_k\sigma_j)$ is antisymmetric in the indices j and k, we replace A_{jik} by its antisymmetric part

$$A_{jik}(\text{anti}) = \tfrac{1}{2}(\gamma_j\gamma_i\gamma_k - \gamma_k\gamma_i\gamma_j) = \gamma_j\delta_{ik} - \gamma_k\delta_{ij} - \gamma_j\gamma_k\gamma_i$$

$$\equiv A_j\delta_{ik} - A_k\delta_{ij} - A_{jki} \quad ; \quad j \neq k \tag{II.29}$$

where
$$\vec{A} \equiv \bar{u}(\vec{p}_1)\vec{\gamma}\, u^*(\vec{p}_2) \tag{II.30}$$

Thus we get

$$Y_1 = 2g_A^2\eta\ (A_j\delta_{ik} - A_k\delta_{ij} - A_{jki})e_{tjk}(\vec{\sigma} \times \vec{\sigma})_t\ r_i$$

$$= 2g_A^2\eta\ e_{tjk}\left\{2A_j\left[r_k(\vec{\sigma} \times \vec{\sigma})_t\right] - A_{jki}\left[r_i(\vec{\sigma} \times \vec{\sigma})_t\right]\right\} \qquad \text{(II.31)}$$

and we see that no scalar can be formed from the nucleon operators, because of the presence of the antisymmetric tensor e_{tjk}.

The second term in (II.23) is

$$Y_2 = -i\ \vec{\ell}'_{\lambda\mu}\langle N_f|\sum_{n,m}(\Gamma_\lambda)_n(\Gamma_\mu)_m\left\{(\vec{p}_1+\vec{p}_2)\cdot(\vec{r}_n+\vec{r}_m)\right\}(\vec{r}_n-\vec{r}_m)|N_i\rangle \qquad \text{(II.32)}$$

$$= -\frac{i}{2}\ \vec{\ell}'_{\lambda\mu}\langle N_f|\sum_{n,m}\left[(\Gamma_\lambda)_n(\Gamma_\mu)_m - (\Gamma_\mu)_n(\Gamma_\lambda)_m\right](\vec{p}_1+\vec{p}_2)\cdot(\vec{r}_n+\vec{r}_m)(\vec{r}_n-\vec{r}_m)|N_i\rangle$$

$$\text{(II.33)}$$

and we write this symbolically as

$$Y_2 = +i\eta\ A_{\lambda i\mu}\left[(\Gamma_\lambda)_n(\Gamma_\mu)_m - (\Gamma_\mu)_n(\Gamma_\lambda)_m\right] P_a R_a r_i \qquad \text{(II.34)}$$

Using the same procedure as above, (II.34) reduces to

$$Y_2 = -ig_A^2\eta\left[A_j\delta_{ik} - A_k\delta_{ij} - A_{jki}\right] P_a\ e_{tjk}\ (\vec{\sigma} \times \vec{\sigma})_t\ r_i R_a$$

$$= -2ig_A^2\eta\left[A_j P_a\ e_{tjk}\ r_k R_a\ (\vec{\sigma} \times \vec{\sigma})_t\right] +$$

$$+\ ig_A^2\eta\left\{A_{jki}\ P_a\ e_{tjk}\ r_i R_a\ (\vec{\sigma} \times \vec{\sigma})_t\right\} \qquad \text{(II.35)}$$

The expression in square brackets in (II.35) is the scalar product of the lepton second rank tensor $A_j P_a$ and the nucleon second rank tensor

$e_{tjk}\ r_k R_a(\vec{\sigma} \times \vec{\sigma})_t$. Expanding these in irreducible tensor representations, we have

$$A_j P_a = \frac{1}{2}\left(A_j P_a + A_a P_j - \frac{2}{3}\,\delta_{ja}\vec{A}\cdot\vec{P}\right) + \frac{1}{2}\left(A_j P_a - A_a P_j\right) + \frac{1}{3}\,\delta_{ja}\,\vec{A}\cdot\vec{P}$$

$$= \frac{1}{2}\,Q_{ja} + \frac{1}{2}\,e_{lja}(\vec{A} \times \vec{P})_l + \frac{1}{3}\,\delta_{ja}\,\vec{A}\cdot\vec{P} \tag{II.36}$$

where $$Q_{ja} \equiv A_j P_a + A_a P_j - \frac{2}{3}\,\delta_{ja}\,\vec{A}\cdot\vec{P} \tag{II.37}$$

and $$r_k R_a = \frac{1}{2}\,S_{ka} + \frac{1}{2}\,e_{uka}(\vec{r} \times \vec{R})_u + \frac{1}{3}\,\delta_{ka}\,\vec{r}\cdot\vec{R} \tag{II.38}$$

where $$S_{ka} \equiv r_k R_a + r_a R_k - \frac{2}{3}\,\delta_{ka}\,\vec{r}\cdot\vec{R} \tag{II.39}$$

Thus the expression in square brackets may be written as

$$\left[\frac{1}{2}\,Q_{ja} + \frac{1}{2}\,e_{lja}(\vec{A} \times \vec{P})_l + \frac{1}{3}\,\delta_{ja}\,\vec{A}\cdot\vec{P}\right] e_{tjk}(\vec{\sigma} \times \vec{\sigma})_t \;\times$$

$$\times \left[\frac{1}{2}\,S_{ka} + \frac{1}{2}\,e_{uka}(\vec{r} \times \vec{R})_u + \frac{1}{3}\,\delta_{ka}\,\vec{r}\cdot\vec{R}\right] \tag{II.40}$$

Now since the expression (II.40) is the scalar product of lepton and nucleon second rank tensors, it follows that if we wish to extract a nucleon scalar, this can be combined only with a lepton scalar, i.e. with $\vec{A}\cdot\vec{P}$. It should be observed that S_{ka} is symmetric ($S_{ka} = S_{ak}$), whereas e_{tjk} is antisymmetric in all pairs of indices. Hence

$$\delta_{ja}(\vec{A}\cdot\vec{P})\;e_{tjk}(\vec{\sigma} \times \vec{\sigma})_t\;S_{ka} = (\vec{A}\cdot\vec{P})(\vec{\sigma} \times \vec{\sigma})_t\;e_{tjk}\;S_{jk} \equiv 0 \tag{II.41}$$

Similarly,

$$\delta_{ja}(\vec{A}\cdot\vec{P})\;e_{tjk}(\vec{\sigma} \times \vec{\sigma})_t\;\delta_{ka}(\vec{r}\cdot\vec{R}) \equiv 0 \tag{II.42}$$

Thus the only nucleon scalar in (II.40) is

$$\frac{1}{6}\,\delta_{ja}(\vec{A}\cdot\vec{P})\;e_{tjk}(\vec{\sigma}\times\vec{\sigma})_t\;e_{uka}(\vec{r}\times\vec{R})_u\;=$$

$$=\;\frac{1}{6}(\vec{A}\cdot\vec{P})\;\delta_{ja}(\delta_{ta}\delta_{ju}-\delta_{tu}\delta_{ja})(\vec{\sigma}\times\vec{\sigma})_t(\vec{r}\times\vec{R})_u$$

where we have used

$$e_{tjk}e_{uka}\;=\;e_{ktj}e_{kau}\;=\;\delta_{ta}\delta_{ju}-\delta_{tu}\delta_{ja} \qquad \text{(II.43)}$$

Thus the contribution from (II.40) is

$$\frac{1}{6}(\vec{A}\cdot\vec{P})(-2\delta_{tu})(\vec{\sigma}\times\vec{\sigma})_t(\vec{r}\times\vec{R})_u\;=\;-\frac{1}{3}(\vec{A}\cdot\vec{P})\Big[(\vec{\sigma}\times\vec{\sigma})\;(\vec{r}\times\vec{R})\Big]$$

$$\equiv\;-\frac{1}{3}\Big[\bar{u}(\vec{p}_1)\vec{\gamma}\cdot\vec{p}\;u^*(\vec{p}_2)\Big]\langle N_f|\sum_{n,m}\tau_n^{(+)}\tau_m^{(+)}(\vec{\sigma}_n\times\vec{\sigma}_m)\cdot\Big[(\vec{r}_n-\vec{r}_m)\times(\vec{r}_n+\vec{r}_m)\Big]|N_i\rangle \qquad \text{(II.44)}$$

Now because of the isospin operators $\tau_n^{(+)}\tau_m^{(+)}$ the labels n and m can be taken to refer to a pair of neutrons in the initial state which is transformed into a pair of protons in the final state. The T = 1 and T = 0 isospin states of a pair of nucleons are given by

$$|1,1\rangle\;=\;|p\rangle\;|p\rangle \qquad \text{(II.45a)}$$

$$|1,0\rangle\;=\;\frac{1}{\sqrt{2}}\Big[|p\rangle\;|n\rangle+|n\rangle\;|p\rangle\Big] \qquad \text{(II.45b)}$$

$$|1,-1\rangle\;=\;|n\rangle\;|n\rangle \qquad \text{(II.45c)}$$

$$|0,0\rangle\;=\;\frac{1}{\sqrt{2}}\Big[|p\rangle\;|n\rangle-|n\rangle\;|p\rangle\Big] \qquad \text{(II.45d)}$$

and we see from (II.45a) and (II.45c) that a pair of protons and a pair of neutrons are both in isospin triplet states, which are symmetric under particle interchange. If, as assumed in the text, pairs of like nucleons are in singlet spin states (antisymmetric under particle interchange), then it follows from the generalized Pauli principle (the total wave function for fermions is antisymmetric under the combined interchange of coordinate, spin and isospin labels) that the space wave function of both the initial and final nuclear states must be symmetric. Thus both the initial and final nuclear states may be written in the form

$$| \ \rangle = |\text{isospin}\rangle_{\text{sym}} \, |\text{spin}\rangle_{\text{anti}} \, |\text{space}\rangle_{\text{sym}}$$

and it follows that the nuclear matrix element

$$\langle N_f | \sum_{n,m} \tau_n^{(+)} \tau_m^{(+)} (\vec{\sigma}_n \times \vec{\sigma}_m) \cdot \left[(\vec{r}_n - \vec{r}_m) \times (\vec{r}_n + \vec{r}_m) \right] | N_i \rangle$$

vanishes, as both the spin and space operators are antisymmetric under interchange of the labels n and m. In other words, the expression (II.44) reduces to zero. It will be noted that in the nuclear matrix elements introduced so far the spin and space operators are symmetric under interchange of the labels n and m.

Going back now to equation (II.35), the expression in curly brackets is

$$A_{jki} \, P_a \, e_{tjk} \, r_i R_a (\vec{\sigma} \times \vec{\sigma})_t$$

$$= A_{jki} \, P_a \, e_{tjk} \left[\frac{1}{2} S_{ia} + \frac{1}{2} e_{uia} (\vec{r} \times \vec{R})_u + \frac{1}{3} \delta_{ia} (\vec{r} \cdot \vec{R}) \right] (\vec{\sigma} \times \vec{\sigma})_t \quad \text{(II.46)}$$

which is the scalar product of the fourth rank lepton tensor $A_{jki}\, P_a$ and a sum of nucleon fourth rank tensors. As stated before, a nucleon scalar must be combined with a lepton scalar to produce an overall scalar, but the presence of the e_{tjk} ensures that the indices j and k cannot be contracted.

We now return to the expression (II.23) and consider the term

$$Y_3 = - i\, \vec{\ell}\,'_{\lambda\mu} \cdot \langle N_f | \sum_{n,m} \cdot\cdot (\Gamma_\lambda)_n (\Gamma_\mu)_m \left\{ (\vec{p}_1 - \vec{p}_2)\cdot(\vec{r}_n - \vec{r}_m) \right\} (\vec{r}_n - \vec{r}_m) | N_i \rangle \qquad \text{(II.47)}$$

$$= - \frac{i}{2}\, \vec{\ell}\,'_{\lambda\mu} \cdot \langle N_f | \sum_{n,m} \cdot\cdot \left[(\Gamma_\lambda)_n (\Gamma_\mu)_m + (\Gamma_\mu)_n (\Gamma_\lambda)_m \right] (p_a r_a)\, \vec{r} | N_i \rangle \qquad \text{(II.48)}$$

which we shall write symbolically as

$$Y_3 = i\eta\, \gamma_\lambda \gamma_i \gamma_\mu\, p_a \left[(\Gamma_\lambda)_n (\Gamma_\mu)_m + (\Gamma_\mu)_n (\Gamma_\lambda)_m \right] r_i r_a \qquad \text{(II.49)}$$

where $$\gamma_\lambda \gamma_i \gamma_\mu \equiv \bar{u}(\vec{p}_1) \gamma_\lambda \gamma_i \gamma_\mu\, u^*(\vec{p}_2)$$

As before, we replace $\gamma_\lambda \gamma_i \gamma_\mu$ by its symmetric part

$$\frac{1}{2}(\gamma_\lambda \gamma_i \gamma_\mu + \gamma_\mu \gamma_i \gamma_\lambda).$$

Also, we do not perform an unrestricted summation over the indices λ and μ, but include only the cases $\lambda = 4$; $\mu = 4$ and $\lambda = j$; $\mu = k$. Recalling that

$$\frac{1}{2}(\gamma_j \gamma_i \gamma_k + \gamma_k \gamma_i \gamma_j) = \gamma_j \delta_{ik} + \gamma_k \delta_{ij} - \gamma_i \delta_{jk}$$

$$\equiv A_j \delta_{ik} + A_k \delta_{ij} - A_i \delta_{jk} \qquad \text{(II.50)}$$

we get from (II.49)

$$Y_3 = -2i\eta g_V^2 A_i p_a r_i r_a + i\eta g_A^2 A_i p_a \delta_{jk}(\sigma_j\sigma_k + \sigma_k\sigma_j) r_i r_a -$$
$$- i\eta g_A^2 (A_j\delta_{ik} + A_k\delta_{ij}) p_a (\sigma_j\sigma_k + \sigma_k\sigma_j) r_i r_a \quad \text{(II.51)}$$

Expanding the lepton and nucleon second rank tensors in irreducible representations, we get

$$A_i p_a = \frac{1}{2} Q_{ia} + \frac{1}{2} e_{tia}(\vec{A} \times \vec{p})_t + \frac{1}{3} \delta_{ia}(\vec{A}\cdot\vec{p}) \quad \text{(II.52)}$$

$$r_i r_a = S_{ia} + \frac{1}{3} \delta_{ia} r^2 \quad \text{(II.53)}$$

$$\sigma_j\sigma_k + \sigma_k\sigma_j = T_{jk} + \frac{2}{3} \delta_{jk}(\vec{\sigma}\cdot\vec{\sigma}) \quad \text{(II.54)}$$

where

$$Q_{ia} \equiv A_i p_a + A_a p_i - \frac{2}{3} \delta_{ia}(\vec{A}\cdot\vec{p}) \quad \text{(II.55)}$$

$$S_{ia} \equiv r_i r_a - \frac{1}{3} \delta_{ia} r^2 \quad \text{(II.56)}$$

$$T_{jk} \equiv \sigma_j\sigma_k + \sigma_k\sigma_j - \frac{2}{3} \delta_{jk}(\vec{\sigma}\cdot\vec{\sigma}) \quad \text{(II.57)}$$

To begin with, we consider the first term in (II.51). Since $r_i r_a$ is symmetric in the indices i and a, we shall replace $A_i p_a$ by its symmetric part, to give

$$-2i\eta\, g_V^2 \left[\frac{1}{2} Q_{ia} + \frac{1}{3} \delta_{ia}(\vec{A}\cdot\vec{p})\right] \left[S_{ia} + \frac{1}{3} \delta_{ia} r^2\right] \quad \text{(II.58)}$$

We note at once that $S_{aa} \equiv 0$, so the only available nucleon scalar is

$$-2i\eta\, g_V^2 + \frac{1}{9} \delta_{ia}\delta_{ia}(\vec{A}\cdot\vec{p})\, r^2 = -\frac{2}{3} i\eta\, g_V^2 (\vec{A}\cdot\vec{p})\, r^2$$

$$\equiv -\frac{1}{3}(2\eta)g_V^2\left[\bar{u}(\vec{p}_1)i\vec{\gamma}\cdot\vec{p}\;u^*(\vec{p}_2)\right]\langle N_f|\sum_{n,m}\tau_n^{(+)}\tau_m^{(+)}(\vec{r}_n-\vec{r}_m)^2|N_i\rangle \tag{II.59}$$

The second term in (II.51) may be written as

$$2i\eta\; g_A^2\left[\frac{1}{2}Q_{ia}+\frac{1}{3}\delta_{ia}(\vec{A}\cdot\vec{p})\right](\vec{\sigma}\cdot\vec{\sigma})\left[S_{ia}+\frac{1}{3}\delta_{ia}\,r^2\right] \tag{II.60}$$

and the only available scalar is

$$\frac{2}{3}\,i\eta\; g_A^2(\vec{A}\cdot\vec{p})(\vec{\sigma}\cdot\vec{\sigma})r^2$$

$$\equiv \frac{1}{3}g_A^2(2\eta)\left[\bar{u}(\vec{p}_1)i\vec{\gamma}\cdot\vec{p}\;u^*(\vec{p}_2)\right]\langle N_f|\sum_{n,m}\tau_n^{(+)}\tau_m^{(+)}(\vec{\sigma}_n\cdot\vec{\sigma}_m)(\vec{r}_n-\vec{r}_m)^2|N_i\rangle \tag{II.61}$$

Finally, the last term in (II.51) is

$$-2i\eta\; g_A^2\;A_jp_a\;\delta_{ik}(\sigma_j\sigma_k+\sigma_k\sigma_j)r_ir_a \tag{II.62}$$

since $$A_k\delta_{ij}(\sigma_j\sigma_k+\sigma_k\sigma_j) = A_j\delta_{ik}(\sigma_j\sigma_k+\sigma_k\sigma_j) \tag{II.63}$$

and (II.62) can be written as

$$-2i\eta\; g_A^2\left[\frac{1}{2}Q_{ja}+\frac{1}{2}e_{tja}(\vec{A}\times\vec{p})_t+\frac{1}{3}\delta_{ja}(\vec{A}\cdot\vec{p})\right]\times$$

$$\times\left[T_{jk}+\frac{2}{3}\delta_{jk}(\vec{\sigma}\cdot\vec{\sigma})\right]\left[S_{ka}+\frac{1}{3}\delta_{ka}\,r^2\right] \tag{II.64}$$

We seek only those terms which couple the lepton scalar $\vec{A}\cdot\vec{p}$ to appropriate nuclear scalars, so we extract the expression

$$\frac{1}{3}\delta_{ja}(\vec{A}\cdot\vec{p})\left[T_{jk}+\frac{2}{3}\delta_{jk}(\vec{\sigma}\cdot\vec{\sigma})\right]\left[S_{ka}+\frac{1}{3}\delta_{ka}\,r^2\right] \tag{II.65}$$

$$= \frac{1}{9}(\vec{A}\cdot\vec{p})\left[3T_{jk}S_{jk} + \delta_{jk}T_{jk}r^2 + 2\delta_{ka}(\vec{\sigma}\cdot\vec{\sigma})S_{ka} + 2(\vec{\sigma}\cdot\vec{\sigma})r^2\right] \qquad \text{(II.66)}$$

Noting that
$$\delta_{jk}T_{jk} = T_{jj} \equiv 0$$
$$\delta_{ka}S_{ka} = S_{kk} \equiv 0$$

we obtain from (II.64)

$$-\frac{1}{9}(2i\eta)g_A^2(\vec{A}\cdot\vec{p})\left[3T_{jk}S_{jk} + 2(\vec{\sigma}\cdot\vec{\sigma})r^2\right] \qquad \text{(II.67)}$$

where
$$T_{jk}S_{jk} = \left[\sigma_j\sigma_k + \sigma_k\sigma_j - \frac{2}{3}\delta_{jk}(\vec{\sigma}\cdot\vec{\sigma})\right]\left[r_jr_k - \frac{1}{3}\delta_{jk}r^2\right]$$
$$= (\sigma_jr_j)(\sigma_kr_k) + (\sigma_kr_k)(\sigma_jr_j) - \frac{2}{3}(\vec{\sigma}\cdot\vec{\sigma})r^2 \qquad \text{(II.68)}$$

and (II.67) becomes

$$-\frac{4}{3}i\eta\, g_A^2(\vec{A}\cdot\vec{p})(\vec{\sigma}_n\cdot\vec{r})(\vec{\sigma}_m\cdot\vec{r})$$

$$\equiv -\frac{4}{3}\eta\, g_A^2\left[\bar{u}(\vec{p}_1)i\vec{\gamma}\cdot\vec{p}\, u^*(\vec{p}_2)\right]\langle N_f|\sum_{n,m}\tau_n^{(+)}\tau_m^{(+)}(\vec{\sigma}_n\cdot\vec{r}_{nm})(\vec{\sigma}_m\cdot\vec{r}_{nm})|N_i\rangle \qquad \text{(II.69)}$$

where
$$\vec{r} \equiv \vec{r}_{nm} \equiv \vec{r}_n - \vec{r}_m$$

Now collecting all the terms involving nucleon scalars, in expressions (II.11), (II.59), (II.61) and (II.69), and, where necessary, inserting the isospin operators $\tau_n^{(+)}\tau_m^{(+)}$ and appropriate powers of $|r_n - r_m|^{-1}$, we see from (2.26) that the matrix element for $0^+ \rightarrow 0^+$ neutrinoless transitions is

$$M = \frac{G^2}{8\pi}(E_1 - E_2)(2\eta)\left[\bar{u}(\vec{p}_1)\gamma_4\, u^*(\vec{p}_2)\right] \times$$

$$\times \left\{g_V^2 \langle N_f| \sum_{n,m} \tau_n^{(+)}\tau_m^{(+)} \frac{1}{|\vec{r}_n - \vec{r}_m|}|N_i\rangle + g_A^2 \langle N_f| \sum_{n,m} \tau_n^{(+)}\tau_m^{(+)} \frac{\vec{\sigma}_n \cdot \vec{\sigma}_m}{|\vec{r}_n - \vec{r}_m|}|N_i\rangle\right\} +$$

$$+ \frac{1}{3}\frac{G^2}{8\pi}(2\eta)\left[\bar{u}(\vec{p}_1) i\vec{\gamma}\cdot\vec{p}\, u^*(\vec{p}_2)\right] \times$$

$$\times \left\{g_V^2 \langle N_f| \sum_{n,m} \tau_n^{(+)}\tau_m^{(+)} \frac{1}{|\vec{r}_n - \vec{r}_m|}|N_i\rangle - g_A^2 \langle N_f| \sum_{n,m} \tau_n^{(+)}\tau_m^{(+)} \frac{\vec{\sigma}_n \cdot \vec{\sigma}_m}{|\vec{r}_n - \vec{r}_m|}|N_i\rangle +\right.$$

$$\left. + 2g_A^2 \langle N_f| \sum_{n,m} \tau_n^{(+)}\tau_m^{(+)} \frac{\left[\vec{\sigma}_n\cdot(\vec{r}_n - \vec{r}_m)\right]\left[\vec{\sigma}_m\cdot(\vec{r}_n - \vec{r}_m)\right]}{|\vec{r}_n - \vec{r}_m|^3}|N_i\rangle\right\}$$

$$- \frac{1}{3}\frac{G^2}{8\pi}\frac{1}{\pi}(2\eta)(E_1 + E_2 + 2\delta)\left[\bar{u}(\vec{p}_1) i\vec{\gamma}\cdot\vec{p}\, u^*(\vec{p}_2)\right] \times$$

$$\times \left\{g_V^2 \langle N_f| \sum_{n,m} \tau_n^{(+)}\tau_m^{(+)} |N_i\rangle - g_A^2 \langle N_f| \sum_{n,m} \tau_n^{(+)}\tau_m^{(+)}\, \vec{\sigma}_n\cdot\vec{\sigma}_m|N_i\rangle +\right.$$

$$\left. + 2g_A^2 \langle N_f| \sum_{n,m} \tau_n^{(+)}\tau_m^{(+)} \frac{\left[\vec{\sigma}_n\cdot(\vec{r}_n - \vec{r}_m)\right]\left[\vec{\sigma}_m\cdot(\vec{r}_n - \vec{r}_m)\right]}{|\vec{r}_n - \vec{r}_m|^2}|N_i\rangle\right\} \qquad \text{(II.70)}$$

APPENDIX III

TRACE CALCULATIONS

In this Appendix we evaluate the sum over spins of the various terms which appear in the expression for $\sum_{\text{spins}} | M |^2$, where M refers to the matrix element for the no-neutrino or the two-neutrino process. As we shall see, this involves a number of Trace calculations, and these are performed in some detail in what follows.

Let us first consider the general expression

$$\sum = \sum_{s_1, s_2} |\bar{u}(\vec{p}_1) \; \Gamma \; u(\vec{p}_2)|^2 \tag{III.1}$$

where Γ is an arbitrary 4 x 4 matrix.

$$\sum = \sum_{s_1, s_2} u^\dagger(\vec{p}_2)\Gamma^\dagger\gamma_4 u(\vec{p}_1)\bar{u}(\vec{p}_1)\Gamma u(\vec{p}_2)$$

$$= \sum_{s_1, s_2} \bar{u}_\alpha(\vec{p}_2)(\bar{\Gamma})_{\alpha\beta} u_\beta(\vec{p}_1)\bar{u}_\lambda(\vec{p}_1)(\Gamma)_{\lambda\rho} u_\rho(\vec{p}_2)$$

where

$$\bar{\Gamma} \equiv \gamma_4 \; \Gamma^\dagger \; \gamma_4 \tag{III.2}$$

$$\sum = \sum_{s_1} \left[u_\beta(\vec{p}_1)\bar{u}_\lambda(\vec{p}_1)\right] \Gamma_{\lambda\rho} \sum_{s_2} \left[u_\rho(\vec{p}_2)\bar{u}_\alpha(\vec{p}_2)\right] \bar{\Gamma}_{\alpha\beta} \tag{III.3}$$

Using the normalization $u_r^\dagger u_s = \delta_{rs}$, we have

$$\sum_{s} u(\vec{p})\bar{u}(\vec{p}) = \frac{m - i\gamma.p}{2E} \; ; \; p = (\vec{p}, iE) \tag{III.4}$$

Equation (III.3) then becomes

$$\sum = \mathrm{Tr} \left\{ \left(\frac{m - i\gamma.p_1}{2E_1} \right) \Gamma \left(\frac{m - i\gamma.p_2}{2E_2} \right) \bar{\Gamma} \right\}$$

$$\sum_{s_1, s_2} |\bar{u}(\vec{p}_1)\Gamma\, u(\vec{p}_2)|^2 = \frac{1}{4E_1E_2} \mathrm{Tr} \left\{ \Gamma(m - i\gamma\cdot p_2)\bar{\Gamma}(m - i\gamma\cdot p_1) \right\} \tag{III.5}$$

Taking the complex conjugate of (III.4) gives

$$\sum_{s} u^{*}(\vec{p})u^{*\dagger}(\vec{p})\gamma_4^{*} = \frac{m + i\gamma\cdot p}{2E} \tag{III.6}$$

since in the Majorana representation

$$\vec{\gamma}^{*} = \vec{\gamma} \; ; \; \gamma_4^{*} = - \gamma_4$$

(III.6) then becomes

$$\sum_{s} u^{*}(\vec{p})u^{*\dagger}(\vec{p})\gamma_4 = \sum_{s} u^{*}(\vec{p})\overline{u^{*}}(\vec{p}) = - \frac{m + i\gamma\cdot p}{2E} \tag{III.7}$$

Using (III.4) and (III.7), it is easily shown that

$$\sum_{s_1, s_2} |\bar{u}(\vec{p}_1)\Gamma\, u^{*}(\vec{p}_2)|^2 = - \frac{1}{4E_1E_2} \mathrm{Tr} \left\{ \Gamma(m + i\gamma\cdot p_2)\Gamma(m - i\gamma\cdot p_1) \right\} \tag{III.8}$$

For subsequent Trace calculations we shall need the following results :

$$\mathrm{Tr}(\text{odd number of } \gamma \text{ matrices}) = 0 \tag{III.9a}$$

$$\mathrm{Tr}\ \gamma_5 = 0 \tag{III.9b}$$

$$\mathrm{Tr}\ \gamma_\lambda \gamma_\mu = 4\ \delta_{\lambda\mu} \tag{III.9c}$$

$$\mathrm{Tr}\ \gamma_5 \gamma_\lambda \gamma_\mu = 0 \tag{III.9d}$$

$$\mathrm{Tr}(\gamma_\alpha \gamma_\beta \gamma_\lambda \gamma_\rho) = 4(\delta_{\alpha\beta}\delta_{\lambda\rho} - \delta_{\alpha\lambda}\delta_{\beta\rho} + \delta_{\alpha\rho}\delta_{\beta\lambda}) \tag{III.9e}$$

$$\mathrm{Tr}\ \gamma_5 \gamma_\alpha \gamma_\beta \gamma_\lambda \gamma_\rho = 4\ e_{\alpha\beta\lambda\rho} \tag{III.9f}$$

where $e_{\alpha\beta\lambda\rho}$ is the totally antisymmetric unit fourth rank tensor. We also note the fact that the Trace of a product of γ matrices is unaffected by cyclic permutation of the matrices in either direction.

<u>Summation of Squared Matrix Elements in Neutrinoless $\beta\beta$ Decay</u>

Now let us consider the expression

$$\sum_1 = \sum_{s_1, s_2} |\bar{u}(\vec{p}_1)\gamma_4 u^*(\vec{p}_2)|^2 \tag{III.10}$$

Here $\Gamma = 4$ and $\bar{\Gamma} = 4$, so using (III.8) we get

$$\sum_1 = -\frac{1}{4E_1E_2}\ \mathrm{Tr}\left\{\gamma_4(m - i\gamma\cdot p_1)\gamma_4(m + i\gamma\cdot p_2)\right\} \tag{III.11}$$

It is clear from (III.9a) that the terms in m give zero. Using (III.9e) (III.11) becomes

$$\sum_1 = -\frac{1}{E_1E_2}\left\{m^2 + (p_1)_\alpha (p_2)_\beta (2\delta_{4\alpha}\delta_{4\beta} - \delta_{44}\delta_{\alpha\beta})\right\}$$

$$= -\frac{1}{E_1E_2}\left\{1 - 2E_1E_2 - (\vec{p}_1\cdot\vec{p}_2 - E_1E_2)\right\}$$

where we have set m equal to 1. Thus

$$\sum_{s_1,s_2} |\overline{u}(\vec{p}_1)\gamma_4 u^*(\vec{p}_2)|^2 = \frac{E_1E_2 - 1 + \vec{p}_1\cdot\vec{p}_2}{E_1E_2} \tag{III.12}$$

Now let

$$\sum{}_2 = \sum_{s_1,s_2} |\overline{u}(\vec{p}_1)u^*(\vec{p}_2)|^2 \tag{III.13}$$

Here $\Gamma = \overline{\Gamma} = 1$, and so

$$\sum{}_2 = -\frac{1}{4E_1E_2}\,\mathrm{Tr}\left\{(m - i\gamma\cdot p_1)(m + i\gamma\cdot p_2)\right\}$$

$$= -\frac{1}{E_1E_2}\left\{m^2 + (p_1)_\alpha(p_2)_\beta\delta_{\alpha\beta}\right\}$$

$$= -\frac{1}{E_1E_2}\left\{1 + \vec{p}_1\cdot\vec{p}_2 - E_1E_2\right\} \quad ; \quad m = 1$$

or

$$\sum_{s_1,s_2} |\overline{u}(\vec{p}_1)u^*(\vec{p}_2)|^2 = \frac{E_1E_2 - 1 - \vec{p}_1\cdot\vec{p}_2}{E_1E_2} \tag{III.14}$$

Similarly, it may be shown that

$$\sum_{s_1,s_2}\left[\overline{u}(\vec{p}_1)\gamma_4 u^*(\vec{p}_2)\right]^\dagger\left[\overline{u}(\vec{p}_1)u^*(\vec{p}_2)\right] = \sum_{s_1,s_2}\left[\overline{u}(\vec{p}_1)u^*(\vec{p}_2)\right]^\dagger\left[\overline{u}(\vec{p}_1)\gamma_4 u^*(\vec{p}_2)\right]$$

$$= -\frac{1}{E_1E_2}(E_1 - E_2) \tag{III.15}$$

Summation of Squared Matrix Elements in Two-neutrino $\beta\beta$ Decay

We shall now evaluate the sum $\sum\limits_{\text{ave } \vec{k}_1, \vec{k}_2} |M|^2$, where from (3.22)

$$\sum |M|^2 = g_V^4 \frac{G^4}{4} \sum \left[K^2|P|^2 + L^2|P'|^2 - KL(P^\dagger P' + P'^\dagger P)\right] |M_1|^2 +$$

$$+ \frac{1}{9} g_A^4 \frac{G^4}{4} \sum \left[K^2|Q|^2 + L^2|Q'|^2 - KL(Q^\dagger Q' + Q'^\dagger Q)\right] |M_2|^2 +$$

$$+ g_V^2 g_A^2 \frac{G^4}{12} KL \sum \left[P^\dagger Q' + P'^\dagger Q)M_1^* M_2 + (Q^\dagger P' + Q'^\dagger P)M_2^* M_1\right] -$$

$$- g_V^2 g_A^2 \frac{G^4}{12} KL \sum \left[(P^\dagger Q + P'^\dagger Q')M_1^* M_2 + (Q^\dagger P + Q'^\dagger P')M_2^* M_1\right] \qquad \text{(III.16)}$$

where

$$P = \left[\bar{u}(\vec{p}_1)\gamma_4(1+\gamma_5)u(\vec{k}_1)\right]\left[\bar{u}(\vec{p}_2)\gamma_4(1+\gamma_5)u(\vec{k}_2)\right] \qquad \text{(3.20a)}$$

$$Q = \left[\bar{u}(\vec{p}_1)\vec{\gamma}(1+\gamma_5)u(\vec{k}_1)\right]\cdot\left[\bar{u}(\vec{p}_2)\vec{\gamma}(1+\gamma_5)u(\vec{k}_2)\right] \qquad \text{(3.21a)}$$

$$P' = \left[\bar{u}(\vec{p}_2)\gamma_4(1+\gamma_5)u(\vec{k}_1)\right]\left[\bar{u}(\vec{p}_1)\gamma_4(1+\gamma_5)u(\vec{k}_2)\right] \qquad \text{(3.20b)}$$

$$Q' = \left[\bar{u}(\vec{p}_2)\vec{\gamma}(1+\gamma_5)u(\vec{k}_1)\right]\cdot\left[\bar{u}(\vec{p}_1)\vec{\gamma}(1+\gamma_5)u(\vec{k}_2)\right] \qquad \text{(3.21b)}$$

We start with the expression

$$\sum_{\text{spins}} \left[K^2|P|^2 + L^2|P'|^2 - KL(P^\dagger P' + P'^\dagger P)\right] \qquad \text{(III.17)}$$

From (3.20a) and (III.5) we see that

$$\sum_{\text{spins}} |P|^2 = \frac{1}{4E_1\epsilon_1} \mathrm{Tr}\left\{\gamma_4(1+\gamma_5)(-i\gamma\, k_1)\gamma_4(1+\gamma_5)(m - i\gamma\cdot p_1)\right\} \times$$

$$\times \frac{1}{4E_2\epsilon_2} \mathrm{Tr}\left\{\gamma_4(1+\gamma_5)(-i\gamma\, k_2)\gamma_4(1+\gamma_5)(m - i\gamma\cdot p_2)\right\} \qquad \text{(III.18)}$$

where $\Gamma = \gamma_4(1+\gamma_5)$; $\bar{\Gamma} = \gamma_4\Gamma^\dagger\gamma_4 = \Gamma$

$$p_j = (\vec{p}_j, iE_j) \quad ; \quad k_j = (\vec{k}_j, i\epsilon_j) \quad ; \quad j = 1, 2$$

and we have taken the neutrino mass to be zero. Thus we may write

$$\sum_{\text{spins}} |\, P\,|^2 = \frac{1}{16E_1E_2\epsilon_1\epsilon_2} \, \mathrm{Tr}^{(1)}\mathrm{Tr}^{(2)} \tag{III.19}$$

$$\mathrm{Tr}^{(1)} = \mathrm{Tr}\left\{\gamma_4(1+\gamma_5)(-i\gamma\cdot k_1)\gamma_4(1+\gamma_5)(m - i\gamma\cdot p_1)\right\}$$

$$= 2\,\mathrm{Tr}\left\{\gamma_4(-i\gamma\cdot k_1)\gamma_4(1+\gamma_5)(m - i\gamma\cdot p_1)\right\} \tag{III.20}$$

$$\mathrm{Tr}^{(2)} = 2\,\mathrm{Tr}\left\{\gamma_4(-i\gamma\cdot k_2)\gamma_4(1+\gamma_5)(m - i\gamma\cdot p_2)\right\} \tag{III.21}$$

In (III.20) the terms with γ_5 give zero, as can be seen from (III.9a) and (III.9f). It also follows from (III.9a) that the term in m gives zero, so we get

$$\begin{aligned}\mathrm{Tr}^{(1)} &= -\,2\,\mathrm{Tr}\left\{\gamma_4(\gamma\cdot k_1)\gamma_4(\gamma\cdot p_1)\right\}\\ &= -\,2\,(p_1)_\alpha(k_1)_\beta\,\mathrm{Tr}\left\{\gamma_\alpha\gamma_4\gamma_\beta\gamma_4\right\}\\ &= -\,8\,(p_1)_\alpha(k_1)_\beta\left[2\delta_{4\alpha}\delta_{4\beta} - \delta_{\alpha\beta}\delta_{44}\right]\\ &= 16E_1\epsilon_1 + 8(\vec{p}_1\cdot\vec{k}_1 - E_1\epsilon_1)\\ &= 8(E_1\epsilon_1 + \vec{p}_1\cdot\vec{k}_1)\end{aligned} \tag{III.22}$$

$\mathrm{Tr}^{(2)}$ is obtained from $\mathrm{Tr}^{(1)}$ by making the substitutions $\vec{p}_1 \to \vec{p}_2$ and

$\vec{k}_1 \to \vec{k}_2$, i.e.

$$\mathrm{Tr}^{(2)} = 8(E_2\epsilon_2 + \vec{p}_2 \cdot \vec{k}_2) \qquad \text{(III.23)}$$

and hence from (III.19) we get

$$\sum_{\text{spins}} | P |^2 = 4 \left(1 + \frac{\vec{p}_1 \cdot \vec{k}_1}{E_1 \epsilon_1}\right)\left(1 + \frac{\vec{p}_2 \cdot \vec{k}_2}{E_2 \epsilon_2}\right) \qquad \text{(III.24)}$$

In averaging over the directions of the neutrino momenta $\vec{k}_j (j = 1, 2)$ only the term 4 survives, and therefore

$$\sum_{\text{ave } \vec{k}_1, \vec{k}_2} | P |^2 = 4 \qquad \text{(III.25)}$$

$\sum_{\text{spins}} | P' |^2$ is obtained from $\sum_{\text{spins}} | P |^2$ by interchanging $\vec{p}_1$ and $\vec{p}_2$.

Thus we get

$$\sum_{\text{spins}} | P' |^2 = 4 \left(1 + \frac{\vec{p}_2 \cdot \vec{k}_1}{E_2 \epsilon_1}\right)\left(1 + \frac{\vec{p}_1 \cdot \vec{k}_2}{E_1 \epsilon_2}\right) \qquad \text{(III.26)}$$

and

$$\sum_{\text{ave } \vec{k}_1, \vec{k}_2} |P'|^2 = 4 \qquad \text{(III.27)}$$

Let us now examine the quantity $\sum_{\text{spins}} P^\dagger P'$, where

$$P = \left[\overline{u}(\vec{p}_1) \Gamma\, u(\vec{k}_1)\right]\left[\overline{u}(\vec{p}_2) \Gamma\, u(\vec{k}_2)\right]$$

$$P' = \left[\overline{u}(\vec{p}_2) \Gamma\, u(\vec{k}_1)\right]\left[\overline{u}(\vec{p}_1) \Gamma\, u(\vec{k}_2)\right]$$

with $$\Gamma = \gamma_4(1 + \gamma_5) = \bar{\Gamma}$$

$$\sum_{\text{spins}} P^{\dagger} P' = \mathrm{Tr}\left\{\Gamma\left(\frac{m - i\gamma\cdot p_1}{2E_1}\right)\Gamma\left(-\frac{i\gamma\cdot k_2}{2\epsilon_2}\right)\Gamma\left(\frac{m - i\gamma\cdot p_2}{2E_2}\right)\Gamma\left(-\frac{i\gamma\cdot k_1}{2\epsilon_1}\right)\right\}$$

$$= -\frac{1}{16E_1E_2\epsilon_1\epsilon_2}\,\mathrm{Tr}\left\{\left[\gamma_4(1 + \gamma_5)(\gamma\cdot k_1)\gamma_4(1 + \gamma_5)(m - i\gamma\cdot p_1)\right] \times\right.$$

$$\left.\times\left[\gamma_4(1 + \gamma_5)(\gamma\cdot k_2)\gamma_4(1 + \gamma_5)(m - i\gamma\cdot p_2)\right]\right\} \qquad \text{(III.28)}$$

$$= \frac{-1}{4E_1E_2\epsilon_1\epsilon_2}\mathrm{Tr}\left\{\gamma_4(\gamma\cdot k_1)\gamma_4(1 + \gamma_5)(m - i\gamma\cdot p_1)\gamma_4(\gamma\cdot k_2)\gamma_4(1 + \gamma_5)(m - i\gamma\cdot p_2)\right\}$$

On examining the Trace in (III.28) we see from (III.9a) that the terms in m give zero. The term in m^2 is

$$m^2\,\mathrm{Tr}\left\{\ldots.(1 + \gamma_5)\gamma_4(\gamma\cdot k_2)\gamma_4(1 + \gamma_5)\right\} =$$

$$= m^2\,\mathrm{Tr}\left\{\ldots.\gamma_4(\gamma\cdot k_2)\gamma_4(1 - \gamma_5)(1 + \gamma_5)\right\} \equiv 0$$

Therefore $\sum_{\text{spins}} P^{\dagger} P' =$

$$= \frac{1}{4E_1E_2\epsilon_1\epsilon_2}\,\mathrm{Tr}\left\{\gamma_4(\gamma\cdot k_1)\gamma_4(1 + \gamma_5)(\gamma\cdot p_1)\gamma_4(\gamma\cdot k_2)\gamma_4(1 + \gamma_5)(\gamma\cdot p_2)\right\}$$

$$= \frac{1}{2E_1E_2\epsilon_1\epsilon_2}\,\mathrm{Tr}\left\{\gamma_4(\gamma\cdot k_1)\gamma_4(\gamma\cdot p_1)\gamma_4(\gamma\cdot k_2)\gamma_4(1 + \gamma_5)(\gamma\cdot p_2)\right\}$$

$$= \frac{1}{2E_1E_2\epsilon_1\epsilon_2}\,(k_1)_\alpha(p_1)_\beta(k_2)_\lambda(p_2)_\rho \quad \times$$

$$\times\left[\mathrm{Tr}\left\{\gamma_\alpha\gamma_4\gamma_\beta\gamma_4\gamma_\lambda\gamma_4\gamma_\rho\gamma_4\gamma_5\right\} + \mathrm{Tr}\left\{\gamma_\alpha\gamma_4\gamma_\beta\gamma_4\gamma_\lambda\gamma_4\gamma_\rho\gamma_4\right\}\right] \qquad \text{(III.29)}$$

Using the anticommutation relation $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\,\delta_{\mu\nu}$ and the equations (III.9), we get

$$\sum_{\text{spins}} P^{\dagger} P' = \frac{2}{E_1 E_2 \epsilon_1 \epsilon_2} (k_1)_\alpha (p_1)_\beta (k_2)_\lambda (p_2)_\rho \times$$

$$\Big[4\,\delta_{\beta 4}\delta_{\lambda 4}(2\,\delta_{\alpha 4}\delta_{\rho 4} - \delta_{\alpha\rho}\delta_{44}) - 2\,\delta_{\beta 4}(\delta_{\alpha\lambda}\delta_{\rho 4} - \delta_{\alpha\rho}\delta_{\lambda 4} + \delta_{\alpha 4}\delta_{\lambda\rho})$$

$$- 2\,\delta_{\rho 4}(\delta_{\alpha\beta}\delta_{\lambda 4} - \delta_{\alpha\lambda}\delta_{\beta 4} + \delta_{\alpha 4}\delta_{\beta\lambda}) + (\delta_{\alpha\beta}\delta_{\lambda\rho} - \delta_{\alpha\lambda}\delta_{\beta\rho} + \delta_{\alpha\rho}\delta_{\beta\lambda})$$

$$- 2\,\delta_{\beta 4}\, e_{\alpha 4\lambda\rho} - 2\,\delta_{\rho 4}\, e_{\alpha\beta\lambda 4} + e_{\alpha\beta\lambda\rho}\Big] \qquad \text{(III.30)}$$

$$= \Big\{4\,(p_1)_4(k_2)_4\big[2(k_1)_4(p_2)_4 - p_2\cdot k_1\big] - 2\,(p_1)_4\big[(k_1\cdot k_2)(p_2)_4 -$$

$$- (p_2\cdot k_1)(k_2)_4 + (k_1)_4(p_2\cdot k_2)\big] - 2\,(p_2)_4\big[(p_1\cdot k_1)(k_2)_4 - (k_1\cdot k_2)(p_1)_4 +$$

$$+ (k_1)_4(p_1\cdot k_2)\big] + \big[(p_1\cdot k_1)(p_2\cdot k_2) - (k_1\cdot k_2)(p_1\cdot p_2) + (p_2\cdot k_1)(p_1\cdot k_2)\big]$$

$$- (k_1)_\alpha (p_1)_\beta (k_2)_\lambda (p_2)_\rho \big[2\,\delta_{\beta 4}\, e_{\alpha 4\lambda\rho} + 2\,\delta_{\rho 4}\, e_{\alpha\beta\lambda 4} - e_{\alpha\beta\lambda\rho}\big]\Big\} \frac{2}{E_1 E_2 \epsilon_1 \epsilon_2}$$

(III.31)

When we take the average over the neutrino directions, the terms involving $\vec{p}_i\cdot\vec{k}_j$ and $\vec{k}_i\cdot\vec{k}_j$ (i, j = 1, 2) all vanish. The only terms inside the curly brackets of (III.31) which survive are of the form $\epsilon_1\epsilon_2\,\vec{p}_1\cdot\vec{p}_2$ or $E_1 E_2 \epsilon_1 \epsilon_2$, i.e. $(k_1)_4(k_2)_4(p_1)_j(p_2)_j$ or $(p_1)_4(p_2)_4(k_1)_4(k_2)_4$. The presence of the antisymmetric tensors in the last three terms of (III.31) then clearly indicates that these terms will also vanish on taking the average over the neutrino directions. Thus the average is to be obtained from the expression

$$\frac{2}{E_1E_2\epsilon_1\epsilon_2}\Big[8\ E_1E_2\epsilon_1\epsilon_2 + 2\ E_1\epsilon_2(p_2\cdot k_1 - E_2\epsilon_1) + 2\ E_1\epsilon_1(p_2\cdot k_2 - E_2\epsilon_2) +$$

$$+ 2\ E_2\epsilon_2(p_1\cdot k_1 - E_1\epsilon_1) + 2\ E_2\epsilon_1(p_1\cdot k_2 - E_1\epsilon_2) + (p_1\cdot k_1 - E_1\epsilon_1)(p_2\cdot k_2 - E_2\epsilon_2)$$

$$- (p_1\cdot p_2 - E_1E_2)(k_1\cdot k_2 - \epsilon_1\epsilon_2) + (p_1\cdot k_2 - E_1\epsilon_2)(p_2\cdot k_1 - E_2\epsilon_1)\Big]$$

whence it follows that

$$\sum_{\text{ave } \vec{k}_1,\vec{k}_2} P^\dagger P' = 2\left(1 + \frac{\vec{p}_1\cdot\vec{p}_2}{E_1E_2}\right) \tag{III.32}$$

Similarly, it may be shown that

$$\sum_{\text{ave } \vec{k}_1,\vec{k}_2} P'^\dagger P = 2\left(1 + \frac{\vec{p}_1\cdot\vec{p}_2}{E_1E_2}\right) \tag{III.33}$$

Thus from (III.17), (III.25), (III.27), (III.32) and (III.33) we obtain

$$\sum_{\text{ave } \vec{k}_1,\vec{k}_2}\Big[K^2\ |P|^2 + L^2\ |P'|^2 - KL(P^\dagger P' + P'^\dagger P)\Big] = 4\Big[K^2 + L^2 - KL\Big(1 + \frac{\vec{p}_1\cdot\vec{p}_2}{E_1E_2}\Big)\Big] \tag{III.34}$$

We now examine the expression

$$\sum_{\text{spins}}\Big[K^2\ |Q|^2 + L^2\ |Q'|^2 - KL(Q^\dagger Q' + Q'^\dagger Q)\Big] \tag{III.35}$$

where $$Q = \Big[\bar{u}(\vec{p}_1)\vec{\gamma}(1 + \gamma_5)u(\vec{k}_1)\Big]\cdot\Big[\bar{u}(\vec{p}_2)\vec{\gamma}(1 + \gamma_5)u(\vec{k}_2)\Big] \tag{3.21a}$$

$$Q' = \left[\overline{u}(\vec{p}_2)\vec{\gamma}(1 + \gamma_5)u(\vec{k}_1)\right]\cdot\left[\overline{u}(\vec{p}_1)\vec{\gamma}(1 + \gamma_5)u(\vec{k}_2)\right] \qquad (3.21b)$$

$$\sum_{\text{spins}} | Q |^2 = \mathrm{Tr}\left\{\gamma_j(1 + \gamma_5)\left(i\frac{\gamma\cdot k_1}{2\epsilon_1}\right)(1 - \gamma_5)\gamma_i\left(\frac{m - i\gamma\cdot p_1}{2E_1}\right)\right\} \times$$

$$\times\ \mathrm{Tr}\left\{\gamma_j(1 + \gamma_5)\left(i\ \frac{\gamma\cdot k_2}{2\epsilon_2}\right)(1 - \gamma_5)\gamma_i\left(\ \frac{m - i\gamma\cdot p_2}{2E_2}\right)\right\} \qquad (III.36)$$

where $\Gamma = \gamma_j(1 + \gamma_5)$

$$\overline{\Gamma} = \gamma_4\Gamma^\dagger\gamma_4 = \gamma_4(1 + \gamma_5)\gamma_j\gamma_4 = -(1 - \gamma_5)\gamma_j$$

Thus
$$\sum_{\text{spins}} | Q |^2 = -\frac{1}{4E_1E_2\epsilon_1\epsilon_2}\ \mathrm{Tr}^{(3)}\mathrm{Tr}^{(4)} \qquad (III.37)$$

where
$$\mathrm{Tr}^{(3)} = \mathrm{Tr}\left\{\gamma_j(\gamma\cdot k_1)(1 - \gamma_5)\gamma_i(m - i\gamma\cdot p_1)\right\} \qquad (III.38)$$

$$\mathrm{Tr}^{(4)} = \mathrm{Tr}\left\{\gamma_j(\gamma\cdot k_2)(1 - \gamma_5)\gamma_i(m - i\gamma\cdot p_2)\right\} \qquad (III.39)$$

Using (III.9a) we see that the term in m in (III.38) gives zero, so we get

$$\mathrm{Tr}^{(3)} = -i(k_1)_\alpha(p_1)_\beta\ \mathrm{Tr}\left\{\gamma_j\gamma_\alpha(1 - \gamma_5)\gamma_i\gamma_\beta\right\}$$

$$= -i(k_1)_\alpha(p_1)_\beta\left[\mathrm{Tr}\left\{\gamma_j\gamma_\alpha\gamma_i\gamma_\beta\right\} - \mathrm{Tr}\left\{\gamma_5\gamma_i\gamma_\beta\gamma_j\gamma_\alpha\right\}\right]$$

$$= -4i(k_1)_\alpha(p_1)_\beta\left[\left(\delta_{j\alpha}\delta_{i\beta} - \delta_{ji}\delta_{\alpha\beta} + \delta_{j\beta}\delta_{\alpha i}\right) - e_{i\beta j\alpha}\right]$$

$$= -4i\left[(p_1)_i(k_1)_j - \delta_{ji}(p_1\cdot k_1) + (p_1)_j(k_1)_i - e_{i\beta j\alpha}(p_1)_\beta(k_1)_\alpha\right] \qquad (III.40)$$

$\mathrm{Tr}^{(4)}$ is obtained from $\mathrm{Tr}^{(3)}$ by making the substitutions $k_1 \to k_2$ and

$p_1 \to p_2$. Thus (III.37) reduces to

$$\sum_{\text{spins}} |Q|^2 = \frac{4}{E_1 E_2 \epsilon_1 \epsilon_2} \times$$

$$\times \left[(p_1)_i (k_1)_j - \delta_{ji} (p_1 k_1) + (p_1)_j (k_1)_i - e_{i\beta j\alpha} (p_1)_\beta (k_1)_\alpha \right]$$

$$\times \left[(p_2)_i (k_2)_j - \delta_{ji} (p_2 k_2) + (p_2)_j (k_2)_i - e_{i\lambda j\rho} (p_2)_\lambda (k_2)_\rho \right] \qquad \text{(III.41)}$$

$$= \frac{4}{E_1 E_2 \epsilon_1 \epsilon_2} \Big[2(\vec{p}_1 \cdot \vec{p}_2)(\vec{k}_1 \cdot \vec{k}_2) - 2(\vec{p}_1 \cdot \vec{k}_1)(p_2 k_2) + 2(\vec{p}_1 \cdot \vec{k}_2)(\vec{p}_2 \cdot \vec{k}_1) -$$

$$- 2(\vec{p}_2 \cdot \vec{k}_2)(p_1 k_1) + 3(p_1 k_1)(p_2 k_2) - e_{i\beta j\alpha} (p_1)_\beta (k_1)_\alpha (p_2)_i (k_2)_j -$$

$$- e_{i\beta j\alpha} (p_1)_\beta (k_1)_\alpha (p_2)_j (k_2)_i - e_{i\lambda j\rho} (p_2)_\lambda (k_2)_\rho (p_1)_i (k_1)_j -$$

$$- e_{i\lambda j\rho} (p_2)_\lambda (k_2)_\rho (p_1)_j (k_1)_i + e_{ij\beta\alpha}\, e_{ij\lambda\rho} (p_1)_\beta (p_2)_\lambda (k_1)_\alpha (k_2)_\rho \Big] \qquad \text{(III.42)}$$

In averaging over the neutrino directions terms involving $\vec{p}_a \cdot \vec{k}_b$ and $\vec{k}_a \cdot \vec{k}_b$ (a,b = 1, 2) do not survive, and the only terms in (III.42) we need consider are

$$3(p_1 k_1)(p_2 k_2)$$

and $$e_{ij\beta\alpha}\, e_{ij\lambda\rho} (p_1)_\beta (p_2)_\lambda (k_1)_\alpha (k_2)_\rho$$

Now $$3(p_1 k_1)(p_2 k_2) = 3(\vec{p}_1 \cdot \vec{k}_1 - E_1 \epsilon_1)(\vec{p}_2 \cdot \vec{k}_2 - E_2 \epsilon_2)$$

and therefore contributes an amount $3\, E_1 E_2 \epsilon_1 \epsilon_2$ to the average over the neutrino directions. Recalling that Latin indices (i, j etc.) run from 1 to 3, and Greek indices (α, β, λ, ρ etc.) run from 1 to 4, we have

$$e_{ij\beta\alpha}\, e_{ij\lambda\rho} = \left(e_{ij\beta}\,\delta_{\alpha 4} + e_{ij\alpha}\,\delta_{\beta 4}\right)\left(e_{ij\lambda}\,\delta_{\rho 4} + e_{ij\rho}\,\delta_{\lambda 4}\right) \qquad \text{(III.43)}$$

where $e_{ij\beta}$ is the totally antisymmetric unit third rank tensor. Since β runs from 1 to 4, we must have

$$e_{ij\beta} = e_{ijr}\,\delta_{r\beta}$$

Similarly,

$$e_{ij\lambda} = e_{ijs}\,\delta_{s\lambda}$$

and so

$$\begin{aligned} e_{ij\beta}\, e_{ij\lambda} &= \left(e_{ijr}\, e_{ijs}\right)\delta_{r\beta}\delta_{s\lambda} \\ &= \left(3\,\delta_{rs} - \delta_{jr}\delta_{js}\right)\delta_{r\beta}\delta_{s\lambda} \\ &= 2\,\delta_{r\beta}\delta_{r\lambda} \qquad \text{(III.44)} \end{aligned}$$

where we have used

$$e_{iab}\, e_{icd} = \delta_{ac}\,\delta_{bd} - \delta_{ad}\,\delta_{bc} \qquad \text{(II.43)}$$

Using the same procedure for the other products in (III.43), we get

$$\begin{aligned} e_{ij\beta\alpha}\, e_{ij\lambda\rho} = 2\Big[&\delta_{\alpha 4}\,\delta_{\rho 4}\,\delta_{r\beta}\,\delta_{r\lambda} + \delta_{\alpha 4}\,\delta_{\lambda 4}\,\delta_{r\beta}\,\delta_{r\rho} + \\ &+ \delta_{\beta 4}\,\delta_{\rho 4}\,\delta_{r\alpha}\,\delta_{r\lambda} + \delta_{\beta 4}\,\delta_{\lambda 4}\,\delta_{r\alpha}\,\delta_{r\rho}\Big] \qquad \text{(III.45)} \end{aligned}$$

and so

$$e_{ij\beta\alpha}\, e_{ij\lambda\rho}(p_1)_\beta (p_2)_\lambda (k_1)_\alpha (k_2)_\rho =$$

$$= 2\left[(\vec{p}_1\cdot\vec{p}_2)(-\epsilon_1\epsilon_2) + (\vec{p}_1\cdot\vec{k}_2)(-E_2\epsilon_1) + (\vec{p}_2\cdot\vec{k}_1)(-E_1\epsilon_2) + (\vec{k}_1\cdot\vec{k}_2)(-E_1E_2)\right]$$

(III.46)

and the only term which survives on averaging over the neutrino directions is $-2\epsilon_1\epsilon_2\vec{p}_1\cdot\vec{p}_2$. So we finally obtain

$$\sum_{\text{ave }\vec{k}_1,\vec{k}_2} |Q|^2 = \frac{4}{E_1E_2\epsilon_1\epsilon_2}\left(3E_1E_2\epsilon_1\epsilon_2 - 2\epsilon_1\epsilon_2\,\vec{p}_1\cdot\vec{p}_2\right)$$

$$= 12\left(1 - \frac{2}{3}\frac{\vec{p}_1\cdot\vec{p}_2}{E_1E_2}\right) \qquad \text{(III.47)}$$

Using the same procedure, it is easy to show that

$$\sum_{\text{ave }\vec{k}_1,\vec{k}_2} |Q'|^2 = 12\left(1 - \frac{2}{3}\frac{\vec{p}_1\cdot\vec{p}_2}{E_1E_2}\right) \qquad \text{(III.48)}$$

We must now evaluate $\sum_{\text{ave }\vec{k}_1,\vec{k}_2} (Q^\dagger Q' + Q'^\dagger Q)$, where

$$Q = \left[\bar{u}(\vec{p}_1)\vec{\gamma}(1+\gamma_5)u(\vec{k}_1)\right]\cdot\left[\bar{u}(\vec{p}_2)\vec{\gamma}(1+\gamma_5)u(\vec{k}_2)\right] \qquad \text{(3.21a)}$$

$$Q' = \left[\bar{u}(\vec{p}_2)\vec{\gamma}(1+\gamma_5)u(\vec{k}_1)\right]\cdot\left[\bar{u}(\vec{p}_1)\vec{\gamma}(1+\gamma_5)u(\vec{k}_2)\right] \qquad \text{(3.21b)}$$

Here
$$\Gamma_j = \gamma_j(1+\gamma_5)$$

$$\bar{\Gamma}_j \equiv \gamma_4\Gamma_j^\dagger\gamma_4 = -\gamma_j(1+\gamma_5)$$

$$\sum_{\text{spins}} Q^\dagger Q' = \frac{1}{16E_1E_2\epsilon_1\epsilon_2}\,\mathrm{Tr}\left\{\left[\gamma_j(1+\gamma_5)(m - i\gamma\cdot p_1)\gamma_i(1+\gamma_5)(-i\gamma\cdot k_2)\right]\times\right.$$

$$\left.\times\left[\gamma_j(1+\gamma_5)(m - i\gamma\cdot p_2)\gamma_i(1+\gamma_5)(-i\gamma\cdot k_1)\right]\right\} \qquad \text{(III.49)}$$

Again, the terms in m give zero. Rearranging the terms, we get

$$\sum_{\text{spins}} Q^{\dagger} Q' = \frac{1}{2E_1 E_2 \epsilon_1 \epsilon_2} \mathrm{Tr} \left\{ (\gamma\cdot k_1)\gamma_j(\gamma\cdot p_1)\gamma_i(\gamma\cdot k_2)\gamma_j(\gamma\cdot p_2)\gamma_i(1+\gamma_5) \right\} \tag{III.50}$$

$$= \frac{1}{2E_1 E_2 \epsilon_1 \epsilon_2} (k_1)_\alpha (p_1)_\beta (k_2)_\lambda (p_2)_\rho \left[\mathrm{Tr}^{(5)} + \mathrm{Tr}^{(6)} \right] \tag{III.51}$$

where

$$\mathrm{Tr}^{(5)} = \mathrm{Tr} \left\{ \gamma_5 \gamma_\alpha \gamma_j \gamma_\beta \gamma_i \gamma_\lambda \gamma_j \gamma_\rho \gamma_i \right\} \tag{III.52}$$

$$\mathrm{Tr}^{(6)} = \mathrm{Tr} \left\{ \gamma_\alpha \gamma_j \gamma_\beta \gamma_i \gamma_\lambda \gamma_j \gamma_\rho \gamma_i \right\} \tag{III.53}$$

Using the anticommutation relation $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\,\delta_{\mu\nu}$, we get

$$\begin{aligned} \mathrm{Tr}^{(5)} = {} & 4\,\delta_{j\beta}\,\delta_{j\lambda}\,\mathrm{Tr}\left\{\gamma_5\gamma_\alpha\gamma_i\gamma_\rho\gamma_i\right\} - 4\,\delta_{j\beta}\,\delta_{\alpha i}\,\mathrm{Tr}\left\{\gamma_5\gamma_j\gamma_\lambda\gamma_\rho\gamma_i\right\} + \\ & + 6\,\delta_{j\beta}\,\mathrm{Tr}\left\{\gamma_5\gamma_j\gamma_\lambda\gamma_\rho\gamma_\alpha\right\} - 4\,\delta_{j\lambda}\,\delta_{\rho i}\,\mathrm{Tr}\left\{\gamma_5\gamma_\alpha\gamma_\beta\gamma_j\gamma_i\right\} + \\ & + 6\,\delta_{j\lambda}\,\mathrm{Tr}\left\{\gamma_5\gamma_\alpha\gamma_\beta\gamma_j\gamma_\rho\right\} - 4\,\delta_{\lambda i}\,\delta_{\rho i}\,\mathrm{Tr}\left\{\gamma_5\gamma_\alpha\gamma_\beta\right\} + 2\,\delta_{\lambda i}\,\mathrm{Tr}\left\{\gamma_5\gamma_\alpha\gamma_\beta\gamma_i\gamma_\rho\right\} + \\ & + 2\,\delta_{\rho i}\,\mathrm{Tr}\left\{\gamma_5\gamma_\alpha\gamma_\beta\gamma_\lambda\gamma_i\right\} - 3\,\mathrm{Tr}\left\{\gamma_5\gamma_\alpha\gamma_\beta\gamma_\lambda\gamma_\rho\right\} \end{aligned} \tag{III.54}$$

and using the equations (III.9), the first term in (III.51) is proportional to

$$\begin{aligned} (k_1)_\alpha (p_1)_\beta (k_2)_\lambda (p_2)_\rho \Big[& - 4\,\delta_{j\beta}\,\delta_{\alpha i}\,e_{j\lambda\rho i} + 6\,\delta_{j\beta}\,e_{j\lambda\rho\alpha} - \\ & - 4\,\delta_{j\lambda}\,\delta_{\rho i}\,e_{\alpha\beta j i} + 6\,\delta_{j\lambda}\,e_{\alpha\beta j\rho} + 2\,\delta_{\lambda i}\,e_{\alpha\beta i\rho} + 2\,\delta_{\rho i}\,e_{\alpha\beta\lambda i} - 3\,e_{\alpha\beta\lambda\rho} \Big] \end{aligned} \tag{III.55}$$

and does not contribute to the average over the neutrino directions because of the presence of the antisymmetric fourth rank tensors. Using the same procedure we find that

$$\mathrm{Tr}^{(6)} = \mathrm{Tr}\left\{\left[4\,\delta_{j\beta}\delta_{j\lambda}\gamma_i - 2\,\delta_{j\beta}\gamma_i\gamma_j\gamma_\lambda - 2\,\delta_{j\lambda}\gamma_\beta\gamma_j\gamma_i - \gamma_\beta\gamma_i\gamma_\lambda\right]\gamma_\rho\gamma_i\gamma_\alpha\right\}$$

where

$$\gamma_j\gamma_i\gamma_j = -\gamma_i$$

$$\gamma_j\gamma_\rho\gamma_j = 2\,\delta_{j\rho}\gamma_j - 3\,\gamma_\rho$$

Thus using the equations (III.9), we see that the second term of (III.51) may be written as

$$\frac{2}{E_1E_2\epsilon_1\epsilon_2}(k_1)_\alpha(p_1)_\beta(k_2)_\lambda(p_2)_\rho\Big[\Big(8\,\delta_{j\beta}\,\delta_{j\lambda}\,\delta_{\rho i}\,\delta_{\alpha i} - 12\,\delta_{j\beta}\,\delta_{j\lambda}\,\delta_{\rho\alpha}\Big) +$$

$$-4\,\delta_{j\beta}\,\delta_{\alpha i}\Big(\delta_{ji}\,\delta_{\lambda\rho} - \delta_{i\lambda}\,\delta_{j\rho} + \delta_{i\rho}\,\delta_{j\lambda}\Big) + 6\,\delta_{j\beta}\Big(\delta_{j\alpha}\,\delta_{\lambda\rho} - \delta_{\alpha\lambda}\,\delta_{j\rho} + \delta_{\alpha\rho}\,\delta_{j\lambda}\Big)$$

$$-4\,\delta_{j\lambda}\,\delta_{\rho i}\Big(\delta_{i\alpha}\,\delta_{j\beta} - \delta_{i\beta}\,\delta_{j\alpha} + \delta_{ji}\,\delta_{\alpha\beta}\Big) + 6\,\delta_{j\lambda}\Big(\delta_{\alpha\rho}\,\delta_{j\beta} - \delta_{\beta\rho}\,\delta_{j\alpha} + \delta_{j\rho}\,\delta_{\alpha\beta}\Big)$$

$$-4\,\delta_{\alpha\beta}\,\delta_{i\lambda}\,\delta_{\rho i} + 2\,\delta_{i\lambda}\Big(\delta_{\alpha\beta}\,\delta_{\rho i} - \delta_{i\alpha}\,\delta_{\beta\rho} + \delta_{\alpha\rho}\,\delta_{i\beta}\Big)$$

$$+\,2\,\delta_{\rho i}\Big(\delta_{\alpha\beta}\,\delta_{i\lambda} - \delta_{\alpha\lambda}\,\delta_{i\beta} + \delta_{\alpha i}\,\delta_{\beta\lambda}\Big) - 3\Big[\delta_{\alpha\beta}\,\delta_{\lambda\rho} - \delta_{\alpha\lambda}\,\delta_{\beta\rho} + \delta_{\alpha\rho}\,\delta_{\beta\lambda}\Big)\Big]$$

In averaging over the directions of the neutrinos only terms which are of the form $\epsilon_1\epsilon_2\,\vec{p}_1\cdot\vec{p}_2$ or $\epsilon_1\epsilon_2E_1E_2$ will survive. That is, the terms which contribute to the average are to be obtained from

$$-\frac{2}{E_1E_2\epsilon_1\epsilon_2}(k_1)_\alpha(p_1)_\beta(k_2)_\lambda(p_2)_\rho\ \times$$

$$\times \left[6\, \delta_{j\beta}\, \delta_{\alpha\lambda}\, \delta_{j\rho} + 2\, \delta_{\rho i}\, \delta_{\alpha\lambda}\, \delta_{i\beta} - 3 \left(\delta_{\alpha\beta}\, \delta_{\lambda\rho} - \delta_{\alpha\lambda}\, \delta_{\beta\rho} + \delta_{\alpha\rho}\, \delta_{\beta\lambda} \right) \right]$$

$$= \frac{2}{E_1 E_2 \epsilon_1 \epsilon_2} \left[- 8\, \vec{p}_1 \cdot \vec{p}_2 (\vec{k}_1 \cdot \vec{k}_2 - \epsilon_1 \epsilon_2) - 3(\vec{p}_1 \cdot \vec{k}_1 - E_1 \epsilon_1)(\vec{p}_2 \cdot \vec{k}_2 - E_2 \epsilon_2) + \right.$$

$$\left. + 3(\vec{p}_1 \cdot \vec{p}_2 - E_1 E_2)(\vec{k}_1 \cdot \vec{k}_2 - \epsilon_1 \epsilon_2) - 3(\vec{p}_1 \cdot \vec{k}_2 - E_1 \epsilon_2)(\vec{p}_2 \cdot \vec{k}_1 - E_2 \epsilon_1) \right]$$

Hence
$$\sum_{\text{ave } \vec{k}_1, \vec{k}_2} Q^{\dagger} Q' = \frac{2}{E_1 E_2 \epsilon_1 \epsilon_2} \left(5\, \vec{p}_1 \cdot \vec{p}_2 - 3\, E_1 E_2 \epsilon_1 \epsilon_2 \right)$$

$$= -6 \left(1 - \frac{5}{3} \frac{\vec{p}_1 \cdot \vec{p}_2}{E_1 E_2} \right) \qquad \text{(III.56)}$$

$\sum_{\text{ave } \vec{k}_1, \vec{k}_2} Q'^{\dagger} Q$ is obtained from (III.56) by interchanging $\vec{p}_1$ and $\vec{p}_2$, i.e.

$$\sum_{\text{ave } \vec{k}_1, \vec{k}_2} Q'^{\dagger} Q = -6 \left(1 - \frac{5}{3} \frac{\vec{p}_1 \cdot \vec{p}_2}{E_1 E_2} \right) \qquad \text{(III.57)}$$

Substituting (III.47), (III.48), (III.56) and (III.57) into (III.35), we get

$$\sum_{\text{ave } \vec{k}_1, \vec{k}_2} \left[K^2 |Q|^2 + L^2 |Q'|^2 - KL(Q^{\dagger} Q' + Q'^{\dagger} Q) \right] =$$

$$= 12 \left[(K^2 + L^2) \left(1 - \frac{2}{3} \frac{\vec{p}_1 \cdot \vec{p}_2}{E_1 E_2} \right) + KL \left(1 - \frac{5}{3} \frac{\vec{p}_1 \cdot \vec{p}_2}{E_1 E_2} \right) \right] \qquad \text{(III.58)}$$

Finally, we consider the expression

$$\sum_{\text{spins}} \left[(P^{\dagger} Q' + P'^{\dagger} Q) M_1^* M_2 + (Q^{\dagger} P' + Q'^{\dagger} P) M_2^* M_1 \right] \qquad \text{(III.59)}$$

Using (3.20a) and (3.21b), it is easy to show that

$$\sum_{\text{spins}} P^{\dagger}Q' = \frac{1}{2E_1E_2\epsilon_1\epsilon_2}(k_1)_\alpha(p_1)_\beta(k_2)_\lambda(p_2)_\rho \; Tr\left\{\gamma_\alpha\gamma_4\gamma_\beta\gamma_j\gamma_\lambda\gamma_4\gamma_\rho\gamma_j(1+\gamma_5)\right\} \tag{III.60}$$

As shown earlier, the term with γ_5 does not contribute to the average over the neutrino directions, and will be left out. With this omission, we may write

$$\begin{aligned}\sum P^{\dagger}Q' = \frac{2}{E_1E_2\epsilon_1\epsilon_2}(k_1)_\alpha(p_1)_\beta(k_2)_\lambda(p_2)_\rho \Big[& 4\,\delta_{j\lambda}\,\delta_{j\rho}\big(2\,\delta_{\alpha4}\,\delta_{\beta4} - \delta_{\alpha\beta}\,\delta_{44}\big) \\ & - 4\,\delta_{j\lambda}\,\delta_{\beta4}\big(\delta_{\alpha4}\,\delta_{j\rho} - \delta_{j\alpha}\,\delta_{\rho4}\big) + 2\,\delta_{j\lambda}\big(\delta_{\alpha\beta}\,\delta_{j\rho} - \delta_{j\alpha}\,\delta_{\beta\rho} + \delta_{\alpha\rho}\,\delta_{j\beta}\big) - \\ & - 4\,\delta_{j\rho}\,\delta_{\alpha4}\big(\delta_{\beta4}\,\delta_{j\lambda} - \delta_{\lambda4}\,\delta_{j\beta}\big) + 2\,\delta_{j\rho}\big(\delta_{\alpha\beta}\,\delta_{j\lambda} - \delta_{\alpha\lambda}\,\delta_{j\beta} + \delta_{j\alpha}\,\delta_{\beta\lambda}\big) - \\ & - 12\,\delta_{\beta4}\,\delta_{\lambda4}\,\delta_{\alpha\rho} + 6\,\delta_{\beta4}\big(\delta_{\alpha4}\,\delta_{\lambda\rho} - \delta_{\alpha\lambda}\,\delta_{\rho4} + \delta_{\alpha\rho}\,\delta_{\lambda4}\big) \\ & + 6\,\delta_{\lambda4}\big(\delta_{\alpha\beta}\,\delta_{\rho4} - \delta_{\alpha4}\,\delta_{\beta\rho} + \delta_{\alpha\rho}\,\delta_{\beta4}\big) - 3\big(\delta_{\alpha\beta}\,\delta_{\lambda\rho} - \delta_{\alpha\lambda}\,\delta_{\beta\rho} + \delta_{\alpha\rho}\,\delta_{\beta\lambda}\big)\Big]\end{aligned} \tag{III.61}$$

and extracting only those terms of the form $\epsilon_1\epsilon_2\,\vec{p}_1\cdot\vec{p}_2$ or $\epsilon_1\epsilon_2E_1E_2$, we get

$$\sum_{\text{ave }\vec{k}_1,\vec{k}_2} P^{\dagger}Q' = -6\left(1 - \frac{1}{3}\frac{\vec{p}_1\cdot\vec{p}_2}{E_1E_2}\right) \tag{III.62}$$

By interchanging $\vec{p}_1$ and $\vec{p}_2$, we get

$$\sum_{\text{ave }\vec{k}_1,\vec{k}_2} P'^{\dagger}Q = -6\left(1 - \frac{1}{3}\frac{\vec{p}_1\cdot\vec{p}_2}{E_1E_2}\right) \tag{III.63}$$

Similarly, we can show that

$$\sum_{\text{ave } \vec{k}_1, \vec{k}_2} Q^{\dagger} P' = \sum_{\text{ave } \vec{k}_1, \vec{k}_2} Q'^{\dagger} P = -6 \left(1 - \frac{1}{3} \frac{\vec{p}_1 \cdot \vec{p}_2}{E_1 E_2} \right) \tag{III.64}$$

Collecting the terms in equations (III.62) to (III.64), we get

$$\sum_{\text{ave } \vec{k}_1, \vec{k}_2} \left[(P^{\dagger} Q' + P'^{\dagger} Q) M_1^* M_2 + (Q^{\dagger} P' + Q'^{\dagger} P) M_2^* M_1 \right] =$$

$$= -12 \left(1 - \frac{1}{3} \frac{\vec{p}_1 \cdot \vec{p}_2}{E_1 E_2} \right) \left(2 \text{ Re } M_1^* M_2 \right) \tag{III.65}$$

One can also easily show that

$$\sum_{\text{ave } \vec{k}_1, \vec{k}_2} \left[(P^{\dagger} Q + P'^{\dagger} Q') M_1^* M_2 + (Q^{\dagger} P + Q'^{\dagger} P') M_2^* M_1 \right] = 0 \tag{III.66}$$

Substituting the results (III.34), (III.58), (III.65) and (III.66) into (III.16), we get

$$\sum_{\text{ave } \vec{k}_1, \vec{k}_2} | M |^2 = g_V^4 \, G^4 \left[(K^2 + L^2) - KL \left(1 + \frac{\vec{p}_1 \cdot \vec{p}_2}{E_1 E_2} \right) \right] | M_1 |^2 +$$

$$+ \frac{1}{3} g_A^4 \, G^4 \left[(K^2 + L^2) \left(1 - \frac{2}{3} \frac{\vec{p}_1 \cdot \vec{p}_2}{E_1 E_2} \right) + KL \left(1 - \frac{5}{3} \frac{\vec{p}_1 \cdot \vec{p}_2}{E_1 E_2} \right) \right] | M_2 |^2$$

$$- g_V^2 \, g_A^2 \, G^4 KL \left(1 - \frac{1}{3} \frac{\vec{p}_1 \cdot \vec{p}_2}{E_1 E_2} \right) \left(2 \text{ RE } M_1^* M_2 \right) \tag{III.67}$$

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